

# Generations of correlation averages

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**Abstract.** The present paper is a dissertation on the possible consequences of a conjectural bound for the so-called modified Selberg integral of the divisor function  $d_3$ , i.e. a discrete version of the classical Selberg integral, where  $d_3(n) = \sum_{abc=n} 1$  is attached to the Cesaro weight  $1 - |n - x|/H$  in the short interval  $|n - x| \leq H$ . Mainly, an immediate consequence is a non-trivial bound for the Selberg integral of  $d_3$ , improving recent results of Ivić based on the standard approach through the moments of the Riemann zeta function on the critical line. We proceed instead with elementary arguments, by first applying the “elementary Dispersion Method” in order to establish a link between “weighted Selberg integrals” of any arithmetic function  $f$  and averages of correlations of  $f$  in short intervals. Moreover, we provide a conditional generalization of our results to the analogous problem on the divisor function  $d_k$  for any  $k \geq 3$ . Further, some remarkable consequences on the  $2k$ -th moments of the Riemann zeta function are discussed. Finally, we also discuss the essential properties that a general function  $f$  should satisfy so that the estimation of its Selberg integrals could be approachable by our method.

## 0. Libretto: introduction and statement of the results.

In the milestone paper [S] Selberg introduced a determinant tool in the study of the distribution of prime numbers in *short intervals*  $[x, x + H]$ , i.e.  $H = o(x)$  as  $x \rightarrow \infty$ , namely the integral

$$\int_N^{2N} \left| \sum_{x < n \leq x+H} \Lambda(n) - H \right|^2 dx ,$$

where  $\Lambda$  is the von Mangoldt function defined as  $\Lambda(n) \stackrel{\text{def}}{=} \log p$  if  $n = p^r$  for some prime number  $p$  and for some positive integer  $r$ , otherwise  $\Lambda(n) \stackrel{\text{def}}{=} 0$ . Thus,  $\Lambda$  is a weighted characteristic function of the prime numbers and it is generated by (minus) the logarithmic derivative of the Riemann zeta function, i.e. its Dirichlet series is  $-\zeta'(s)/\zeta(s)$ . Further, being a quadratic mean, the Selberg integral precisely concerns the study of the distribution of primes in *almost all* short intervals  $[x, x + H]$  with at most  $o(N)$  exceptional integers  $x \in [N, 2N]$  as  $N \rightarrow \infty$ . Here we define the *Selberg integral* of any arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{C}$  as

$$J_f(N, H) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x+H} f(n) - M_f(x, H) \right|^2 ,$$

where  $x \sim N$  means  $N < x \leq 2N$  and  $M_f(x, H)$  is the expected *mean value* of  $f$  in short intervals (abbreviated as s.i. mean value). In order to avoid trivialities, one assumes that the length  $H$  of the short interval goes to infinity with  $N$ . In view of non-trivial bounds of such sums, it is easy to realize that the discrete version  $J_\Lambda(N, H)$  is close enough to the original integral introduced by Selberg, so that we feel to be legitimate to use the same symbol for both versions. Such conditions hold for the arithmetic functions we work with and the typical case is the  $k$ -divisor function  $d_k$  for  $k \geq 3$ , where  $d_k(n)$  is the number of ways to write  $n$  as a product of  $k$  positive integer factors (see [C0] and compare §3). Let us denote the Selberg integral of  $d_k$  as

$$J_k(N, H) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x+H} d_k(n) - M_k(x, H) \right|^2$$

with the s.i. mean value of  $d_k$  given by

$$M_k(x, H) \stackrel{\text{def}}{=} H \left( P_{k-1}(\log x) + P'_{k-1}(\log x) \right) ,$$

where  $P_{k-1}$  is the *residual polynomial* of degree  $k - 1$  such that  $P_{k-1}(\log x) \stackrel{\text{def}}{=} \text{Res}_{s=1} \zeta^k(s) x^{s-1} / s$ .

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The first author has proved the lower bound  $NH \log^4 N \ll J_3(N, H)$  for  $H \ll N^{1/3-\varepsilon}$  (see [C0]), while, in an attempt to establish a non trivial upper bound, both the authors have formulated the following conjecture for the so-called *modified Selberg integral* of  $d_3$ ,

$$\tilde{J}_3(N, H) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{0 \leq |n-x| \leq H} \left(1 - \frac{|n-x|}{H}\right) d_3(n) - M_3(x, H) \right|^2,$$

where  $M_3(x, H)$  is the same s.i. mean value of  $J_3(N, H)$ .

CONJECTURE CL. *If  $H \ll N^{1/3}$ , then  $\tilde{J}_3(N, H) \ll NH$ .*

Here and in what follows for convenience we write

$$A(N, H) \ll B(N, H) \text{ whenever } A(N, H) \ll_{\varepsilon} N^{\varepsilon} B(N, H) \quad \forall \varepsilon > 0.$$

Moreover, we adopt a further convention on bounds of the *width* of  $H$ , i.e.  $\theta \stackrel{def}{=} \log H / \log N$  (more in general  $\theta$  is defined by  $x^{\theta} \ll H \ll x^{\theta}$  for  $x \sim N$ ): the inequality  $\theta > \theta_0$  (resp.  $\theta < \theta_0$ ) means that there exists a fixed and absolute constant  $\delta > 0$  such that  $\theta \geq \theta_0 + \delta$  (resp.  $\theta \leq \theta_0 - \delta$ ). In particular,  $0 < \theta < 1$  has to be interpreted as  $\delta \leq \theta \leq 1 - \delta$ .

As a consequence one has the following result.

THEOREM 1. *If Conjecture CL holds, then  $J_3(N, H) \ll NH^{3/2}$ .*

Our theorem is an easy deduction by the general link between the Selberg integral  $J_f(N, H)$  and the corresponding modified one (see §4)

$$\tilde{J}_f(N, H) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{0 \leq |n-x| \leq H} \left(1 - \frac{|n-x|}{H}\right) f(n) - M_f(x, H) \right|^2.$$

Noteworthy Theorem 1 implies an improvement on Ivić's results [Iv2] for  $d_3$  both in the bound and in the “low” range where it is valid: while Ivić's bound is non-trivial for  $N^{1/6+\delta} \leq H \leq N^{1-\delta}$ , i.e. for width  $1/6 < \theta < 1$ , ours is non-trivial for  $0 < \theta \leq 1/3$ , that is in the range  $N^{\delta} \leq H \leq N^{1/3}$ . Further, we think that our estimates can be refined in order to get a better range for the width. We remark that, still assuming Conjecture CL, the first author [C5] has recently derived the better bound

$$J_3(N, H) \ll NH^{6/5}.$$

Needless to say that our study applies to any divisor function  $d_k$ , though the conjectured estimates of the modified Selberg integral,

$$\tilde{J}_k(N, H) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{0 \leq |n-x| \leq H} \left(1 - \frac{|n-x|}{H}\right) d_k(n) - M_k(x, H) \right|^2,$$

become less and less meaningful as  $k$  grows. This is essentially due to the poor state of knowledge about the distribution of  $d_k$  in long intervals, namely the known value of the exponent  $\alpha_k$  such that (compare §3)

$$(*) \quad \sum_{n \leq x} d_k(n) - xP_{k-1}(\log x) \ll x^{\alpha_k}.$$

We generalize Conjecture CL for  $\tilde{J}_k(N, H)$  with  $k > 3$  as it follows.

GENERAL CONJECTURE CL. *Assume that  $\alpha_{k-1} \in [0, 1)$  in (\*) for a fixed integer  $k > 3$ . If  $H \ll N^{1/2}$ , then*

$$\tilde{J}_k(N, H) \ll N^{1-1/k} H^2 + NH(N^{1-4/k} + N^{(1-1/k)\alpha_{k-1}-1/k}).$$

Consequently, setting  $\tilde{\theta}_k \stackrel{\text{def}}{=} \max\left(1 - \frac{4}{k}, \left(1 - \frac{1}{k}\right)\alpha_{k-1} - \frac{1}{k}\right)$ , for every width  $\theta \in (\tilde{\theta}_k, 1/2]$  there exists an  $\varepsilon_1 = \varepsilon_1(\theta, k) > 0$  such that

$$\tilde{J}_k(N, H) \ll N^{1-4\varepsilon_1} H^2.$$

Similarly to the case  $k = 3$ , from the last inequality it follows our second main result.

**THEOREM 2.** *If General Conjecture CL holds, then  $J_k(N, H) \ll N^{1-2\varepsilon_1} H^2$ .*

Again as a consequence one would get an improvement in the low range of  $H$  with respect to the results of Ivić for the mean-square of  $d_k$  in short intervals [Iv2]. In fact, Theorem 1 in [Iv2] holds for  $\theta \in (\theta_k, 1)$  with  $\theta_k$  defined in terms of Carlson's abscissae (see §6). In particular, it holds for  $\theta_4 = 1/4$ ,  $\theta_5 = 11/30$ ,  $\theta_6 = 3/7$ , whereas we get non-trivial estimates for widths  $\tilde{\theta}_k < \theta \leq 1/2$  with  $\tilde{\theta}_4 = 11/128$ ,  $\tilde{\theta}_5 = 1/5$ ,  $\tilde{\theta}_6 = 1/3$ . See §7 for further details, where one finds the so-called *k-folding* method that is at the core of our conjectures.

It is apparent from our study that generally  $\tilde{J}_f(N, H)$  fits the request of “smoothing” the Selberg integral  $J_f(N, H)$ , both in the arithmetic and the harmonic analysis aspects. The arithmetic matter essentially relies on a simple observation going back to the Italian mathematician Cesaro around the end of the 19th century:

$$\sum_{0 \leq |n-x| \leq H} \left(1 - \frac{|n-x|}{H}\right) f(n) = \frac{1}{H} \sum_{h \leq H} \sum_{|n-x| < h} f(n).$$

This is a kind of arithmetic mean of the inner sum in  $J_f(N, H)$  and somehow justifies the appearance of the same mean-value term in the modified Selberg integral. The analytic aspects of such a smoothing process will be better understood after the introduction of the *correlation*  $\mathcal{C}_f(h)$  in §2, where it is showed that Selberg integrals of  $f$  are strictly related to averages of  $\mathcal{C}_f(h)$  in short intervals  $|h| \ll H$ .

The next corollaries testify such an intimate link and also conditionally improve recent results [BBMZ] and [IW] on an additive divisor problem for  $d_k$ . More precisely, they concern the *deviation* of  $d_k$ , i.e.

$$\mathbb{D}_k(N, H) \stackrel{\text{def}}{=} \sum_{h \leq H} \sum_{n \sim N} d_k(n) d_k(n-h) - \frac{1}{H} \sum_{n \sim N} M_k(n, H)^2 = \sum_{h \leq H} \mathcal{C}_k(h) - H \sum_{n \sim N} p_{k-1}(\log n)^2,$$

where  $\mathcal{C}_k(h) \stackrel{\text{def}}{=} \sum_{n \sim N} d_k(n) d_k(n-h)$  is the correlation of  $d_k$  (see §2) and  $p_{k-1}(\log n) \stackrel{\text{def}}{=} M_k(n, H)/H$  is the so-called *logarithmic polynomial* of  $d_k(n)$ . The following further consequence of Theorem 1 is proved in §6.

**COROLLARY 1.** *Let  $N, H$  be positive integers such that  $0 < \theta = \log H / \log N \leq 1/3$ . Then*

$$\mathbb{D}_3(N, H) \ll NH^{3/4} + N^{\alpha_3} H + H^2.$$

Analogously, from Theorem 2 we get a non-trivial estimate for the deviation of  $d_k$  for any  $k > 3$  (see §7).

**COROLLARY 2.** *Under the same hypotheses of Theorem 2 one has, in the same ranges and for the same  $\varepsilon_1$ ,*

$$\mathbb{D}_k(N, H) \ll N^{1-\varepsilon_1} H.$$

The aforementioned link to [BBMZ] and [IW] results is due to the identity (see §3)

$$p_{k-1}(\log x) = P_{k-1}(\log x) + P'_{k-1}(\log x) = \operatorname{Res}_{s=1} \zeta^k(s) x^{s-1},$$

where one has to be acquainted that our notations  $P_{k-1}$  and  $p_{k-1}$  are not consistent with those in [BBMZ] and [IW]. In particular, from equation (3.8) in [IW] it turns out that

$$H \int_N^{2N} p_{k-1}(\log x)^2 dx + O_\varepsilon(N^{1+\varepsilon})$$

is the main term in the formulæ for sums of  $d_k$  correlations established in [BBMZ] when  $k = 3$  and in [IW] for every  $k \geq 3$ . Since it is easily seen that

$$\sum_{n \sim N} p_{k-1}(\log n)^2 - \int_N^{2N} p_{k-1}(\log x)^2 dx \ll 1 ,$$

then for every  $H \ll N$  it follows

$$H \sum_{n \sim N} p_{k-1}(\log n)^2 - H \int_N^{2N} p_{k-1}(\log x)^2 dx \ll N ,$$

revealing that within negligible remainders  $\ll N$  our  $\mathbb{D}_k(N, H)$  is comparable with  $\sum_{h \leq H} \Delta_k(N; h)$ , that is the average of errors for  $d_k$  correlations estimated in [BBMZ] when  $k = 3$  and in [IW] for every  $k \geq 3$ . Thus, Corollary 1 and the best known  $\alpha_3 \leq 43/96$  (Kolesnik, 1981) imply that  $\mathbb{D}_3(N, H) \ll NH^{3/4}$  for  $\theta \leq 1/3$ , which improves [BBMZ] in the low range of short intervals ([IW] bounds are better for  $k = 3$  when  $\theta > 1/2$ ). In fact, their remainders total  $\ll N^{13/12} \sqrt{H}$ , that is worst than  $\ll NH^{3/4}$  when  $\theta < 1/3$ , and are non-trivial only for  $\theta > 1/6$ . Similarly, since Corollary 2 holds in the ranges prescribed by the General Conjecture CL, we get an improvement on the estimation of  $\mathbb{D}_k(N, H)$  in the low range of short intervals with respect to [IW] bounds' non-trivial ranges, though they have better high range, say  $1/2 < \theta < 1$ .

The novelty of our approach is that, though conditionally, it leads to valuable improvements with respect to the analogous achievements obtained via the classical moments of the Riemann zeta function on the critical line  $\Re(s) = 1/2$ , while we think that the conjectures CL on the modified Selberg integrals might be approachable by elementary arguments. On the other side, estimates of  $J_k(N, H)$  have non-trivial consequences on the  $2k$ -th moments of  $\zeta$  (see [C2]) defined as

$$I_k(T) \stackrel{\text{def}}{=} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt .$$

Thus, at the moment we content ourselves in having found an alternative way to pursue possible improvements on the  $2k$ -th moments of  $\zeta$  at least for relatively low values of  $k$ . Indeed, in §8 we take a glance at the effect of hypothetical estimates for Selberg integrals on the  $2k$ -th moments through Theorem 1.1 of [C2], whereas our conjectured  $\tilde{J}_3$  bound provides effectively the best known estimate for the 6-th moment of Riemann  $\zeta$  function after recent [C5]. In particular, in §8 we prove next result, which gives a link between conditional bounds of Selberg integrals  $J_k$  and bounds of  $I_k(T)$ .

**THEOREM 3.** *Let  $k \geq 3$  be fixed. If  $J_k(N, H) \ll N^{1+A} H^{1+B}$  holds for  $H \ll N^{1-2/k}$  and for some constants  $A, B \geq 0$ , then  $I_k(T) \ll T^{1+\frac{k}{2}(A+B)-B}$ .*

Here, beyond the dependence on  $\varepsilon$ , the constant involved in  $\ll$  may depend on  $k$ .

As an immediate consequence of Theorem 3 combined with the results in [C5], namely  $J_3(N, H) \ll NH^{6/5}$  in the range  $H \ll N^{1/3}$ , we get

$$I_3(T) \ll T^{11/10} .$$

This encourages us to follow such a pattern and to pursue non trivial estimates for the modified Selberg integrals of  $d_k$  in the future.

### Plan of the paper.

- §1 Beyond the aforementioned instances, further variations of the Selberg integral can be considered, according to the weight  $w$  suitably attached to an arithmetic function  $f$ . We give a very short introduction to the so-called  $w$ -Selberg integral of  $f$ .
- §2 We introduce the correlations of an arithmetic function  $f$  and of a weight  $w$ . For a wide class of arithmetic functions it is shown through the Dispersion Method that weighted Selberg integrals are strictly related to averages of such correlations (see Lemma 1).

- §3 In an attempt to generalize further our results, we have abstracted the essential properties that an arithmetic function  $f$  has to satisfy so that its Selberg integrals can be approachable by our method. Mainly inspired by the prototype  $d_k$ , we devote this section to the definitions and basic properties of the *essentially bounded, balanced, quasi-constant* and *stable* arithmetic functions. Inevitably in this analysis one finds references to the famous Selberg Class.
- §4 For a real, balanced and essentially bounded function  $f$ , the second and the third generation of correlation averages in short intervals correspond respectively to the Selberg integral and the modified one. Here we exploit further the properties of such functions in long intervals to outline a chain of implications that under suitable conditions from a non-trivial estimate for  $\tilde{J}_f(N, H)$  generates a non-trivial bound for  $J_f(N, H)$ . Applying these implications to the divisor functions one gets immediately Theorems 1 and 2. Such a bound for  $J_f(N, H)$  in turn becomes an effective mean to pursue a “good deviation” of  $f$ , i.e. an error term in the asymptotic formula for the first average of the correlation of  $f$  in short intervals, whereas, as we shortly recall in §5, the expected formula for the single correlation is just conjectural for most significant instances of  $f$ .
- §5 It is a short excursion on some very special cases of single correlations whose conjectured asymptotic formulæ have been proved.
- §6 Here one finds the proof of Corollary 1, that follows rather easily through the general arguments of §4. Corollary 2 follows similarly although it is discussed in the next section.
- §7 At least in principle, the strategy applies also to  $d_k$  for any  $k > 3$  as an application of the general  $k$ -folding method that is described by Lemma 2.
- §8 As an application of Theorem 1.1 in [C2], we prove Theorem 3, that emphasizes the consequences of  $J_k(N, H)$  bounds on the moments of the Riemann zeta function on the critical line.
- §9 We call upon the Selberg integral of  $d_3$  to address the last word on the best “unconditional” exponent for the 6–th moment, that becomes an immediate consequence of Conjecture CL, Theorem 3 and [C5] bounds.

### Some notation and conventions.

If the implicit constants in  $O$  and  $\ll$  symbols depend on some parameters like  $\varepsilon > 0$ , then mostly we specify it by introducing subscripts in such symbols like  $O_\varepsilon$  and  $\ll_\varepsilon$ , while we avoid subscripts for  $\lll$  defined above. Notice that the value of  $\varepsilon$  may change from statement to statement, since  $\varepsilon > 0$  is arbitrarily small.

The relation  $f \sim g$  between the functions  $f, g$  means that  $f = g + o(g)$  as the main variable tends to infinity typically. No confusion should be possible with the *dyadic* notation,  $x \sim N$ , which means that  $x$  is an integer of the interval  $(N, 2N]$ , as already said.

The *Möbius function* is defined as  $\mu(1) = 1$ ,  $\mu(n) = (-1)^r$  if  $n$  is the product of  $r$  distinct primes, and  $\mu(n) = 0$  otherwise. The symbol  $\mathbf{1}$  denotes the constant function with value 1 and  $\mathbf{1}_U$  is the characteristic function of the set  $U$ . The *Dirichlet convolution product* of the arithmetic functions  $f_1$  and  $f_2$  is

$$(f_1 * f_2)(n) \stackrel{\text{def}}{=} \sum_{d|n} f_1(d) f_2(n/d) \quad \forall n \in \mathbb{N}.$$

In particular, we call  $f_k \stackrel{\text{def}}{=} \underbrace{f * \cdots * f}_{k \text{ times}}$  the  $k$ -fold *Dirichlet product* of the arithmetic function  $f$ . For any

$f, g : \mathbb{N} \rightarrow \mathbb{C}$  the *Möbius inversion formula* states that  $f = g * \mathbf{1}$  if and only if  $g = f * \mu$ , which is called the *Eratosthenes transform* of  $f$ . For example,  $\mathbf{1}$  is the Eratosthenes transform of the divisor function  $\mathbf{d} = \mathbf{1} * \mathbf{1}$ . More in general,  $d_k = \underbrace{\mathbf{1} * \cdots * \mathbf{1}}_{k \text{ times}} = \mathbf{1}_k$  is the  $k$ -fold Dirichlet product of  $\mathbf{1}$  for  $k \geq 2$ .

For simplicity, in sums like  $\sum_{a \leq X}$  it is implicit that  $a \geq 1$ . The distance of a real number  $\alpha$  from the nearest integer is denoted by  $\|\alpha\| = \min(\{\alpha\}, 1 - \{\alpha\})$ , where  $\{\alpha\}$  is the fractional part of  $\alpha$ . As usual, we set  $e(\alpha) = e^{2\pi i \alpha}$ ,  $\forall \alpha \in \mathbb{R}$ , and  $e_q(a) = e(a/q)$ ,  $\forall q \in \mathbb{N}$ ,  $\forall a \in \mathbb{Z}$ .

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## 1. Preludio: weighted Selberg integrals.

Given positive integers  $N$  and  $H = o(N)$ , the  $w$ -Selberg integral of an arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{C}$  is the weighted quadratic mean

$$J_{w,f}(N, H) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_n w(n-x) f(n) - M_f(x, w) \right|^2 ,$$

where the complex valued *weight*  $w$  has support in  $[-cH, cH]$  for some fixed real number  $c > 0$ , so that the inner sum is genuinely finite. The term  $M_f(x, w)$  is the expected mean value of  $f$  weighted with  $w$  in the short interval of length  $\ll H$  and its dependence on  $w$  has to be specified according to the different instances. In particular, according to the study exposed in §3, when it is possible to define the *logarithmic polynomial*  $p_f(\log n)$  we set

$$M_f(x, w) \stackrel{\text{def}}{=} \sum_n w(n-x) p_f(\log n) = p_f(\log x) \sum_h w(h) + O(H^2 N^{\varepsilon-1}) .$$

Clearly, the weighted Selberg integrals include the most celebrated case of the original Selberg integral, since  $J_\Lambda(N, H) = J_{u,\Lambda}(N, H)$ , where  $u \stackrel{\text{def}}{=} \mathbf{1}_{[1, H]}$  is the characteristic function of  $[1, H]$ . More in general,  $J_f(N, H)$  is the  $u$ -Selberg integral of  $f$ . The modified Selberg integral  $\tilde{J}_f(N, H)$ , introduced by the first author in [C1], is recognizable as a weighted Selberg integral by taking the *Cesaro weight*, say

$$C_H(t) \stackrel{\text{def}}{=} \left(1 - \frac{|t|}{H}\right)_+ = \max\left(1 - \frac{|t|}{H}, 0\right) .$$

Since

$$C_H(t) = \frac{1}{H} \sum_{a \leq H-|t|} 1 = \frac{1}{H} \sum_{\substack{a, b \leq H \\ b-a=t}} 1 = \frac{\mathcal{C}_u(t)}{H} ,$$

where  $\mathcal{C}_u$  is the correlation of the weight  $u$  (see next §2), then we refer to the Cesaro weight as the “normalized correlation” of  $u$ . More in general, we *smooth* the weighted Selberg integral  $J_{w,f}(N, H)$  by defining the *modified  $w$ -Selberg integral* of  $f$  as

$$\tilde{J}_{w,f}(N, H) \stackrel{\text{def}}{=} J_{\tilde{w},f}(N, H) ,$$

where the new weight  $\tilde{w}$  is the normalized correlation of  $w$ , i.e.

$$\tilde{w}(h) \stackrel{\text{def}}{=} \frac{1}{H} \sum_{\substack{n \\ n-m=h}} \sum_m w(n) \overline{w(m)} .$$

Another important instance of the weighted Selberg integral has been intensively studied by the first author, i.e. the *symmetry integral* of  $f$  given by

$$J_{\text{sgn},f}(N, H) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{0 \leq |n-x| \leq H} \text{sgn}(n-x) f(n) \right|^2 ,$$

where  $\text{sgn}(0) \stackrel{\text{def}}{=} 0$ ,  $\text{sgn}(t) \stackrel{\text{def}}{=} |t|/t$  for  $t \neq 0$ , and  $M_f(x, \text{sgn})$  vanishes identically for every  $f$ . The study of the symmetry integral has been motivated by the work of Kaczorowski and Perelli (see [KP]), who were the very first to exploit a strict relation of the classical Selberg integral with the symmetry properties of the prime numbers. Indeed, in [C], [CS], [C3], [C4], the symmetric aspects of the distribution of several samples of arithmetic functions in short intervals are studied through the analysis of the associated symmetry integral.

It is worthwhile to exploit the link between  $J_{\text{sgn},f}$  and the *modified symmetry integral*  $\tilde{J}_{\text{sgn},f}$  in future papers. As in the case of any odd weight  $w$ , they demand the s.i. mean values to vanish. On the other side, note that the normalized correlation of any  $w$  is even and inside  $M_f(x, \tilde{w})$  we have  $\sum_h \tilde{w}(h) = |\sum_a w(a)|^2 / H$ .

Finally, some considerations in §3 make it plausible that a satisfactory general theory, for the weighted Selberg integrals, may be built within the environment of the Selberg Class.

## 2. Overture: weighted Selberg integrals as correlation averages.

By *correlation* of an arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{C}$  we mean a shifted convolution sum of the form

$$\mathcal{C}_f(h) \stackrel{\text{def}}{=} \sum_{n \sim N} f(n) \overline{f(n-h)},$$

where the *shift*  $h$  is an integer such that  $|h| \leq N$ . Observe that one might take into account only the restriction of  $f$  to  $1 \leq n \leq 3N$ . Further, a correlation of shift  $h$  is essentially a weighted count of the integer solutions  $n, m \in (N, 2N]$  of the equation  $n - m = h$ , as

$$\mathcal{C}_f(h) = \sum_{\substack{n \sim N \\ n-m=h}} \sum_{m \sim N} f(n) \overline{f(m)} + O\left(|h| \cdot \max_{\ell \sim N} |f(\ell)|^2\right).$$

In the present context it is convenient to define the correlation of a weight  $w$  by neglecting the  $O$ -term, i.e.

$$\mathcal{C}_w(h) \stackrel{\text{def}}{=} \sum_{\substack{a \\ b-a=h}} \sum_b w(b) \overline{w(a)}.$$

The reason of such a different definition will be clarified after next lemma, where we prove a strict connection between correlations and weighted Selberg integrals by applying an elementary *Dispersion Method*.

LEMMA 1. *Let  $N, H$  be positive integers such that  $H \rightarrow \infty$  and  $H = o(N)$  as  $N \rightarrow \infty$ . For every uniformly bounded weight  $w$  with support in  $[-cH, cH]$  and every arithmetic function  $f$  one has*

$$J_{w,f}(N, H) = \sum_{0 \leq |h| \leq 2cH} \mathcal{C}_w(h) \mathcal{C}_f(h) - 2\Re \left( \sum_n f(n) \sum_{x \sim N} w(n-x) \overline{M_f(x, w)} \right) + \sum_{x \sim N} |M_f(x, w)|^2 + O(H^3 \|f\|_\infty^2),$$

$$\text{where } \|f\|_\infty \stackrel{\text{def}}{=} \max_{N-cH < n \leq 2N+cH} |f(n)|.$$

PROOF. By expanding the square and exchanging sums one gets

$$\begin{aligned} J_{w,f}(N, H) &= \sum_{x \sim N} \left( \sum_n w(n-x) f(n) - M_f(x, w) \right) \left( \sum_m \overline{w(m-x)} \cdot \overline{f(m)} - \overline{M_f(x, w)} \right) = \\ &= \sum_n f(n) \sum_m \overline{f(m)} \sum_{x \sim N} w(n-x) \overline{w(m-x)} - 2\Re \left( \sum_n f(n) \sum_{x \sim N} w(n-x) \overline{M_f(x, w)} \right) + \sum_{x \sim N} |M_f(x, w)|^2. \end{aligned}$$

Thus, it suffices to show that

$$\sum_n f(n) \sum_m \overline{f(m)} \sum_{x \sim N} w(n-x) \overline{w(m-x)} = \sum_h \mathcal{C}_w(h) \mathcal{C}_f(h) + O(H^3 \|f\|_\infty^2),$$

where we may clearly assume that  $a \stackrel{\text{def}}{=} m-x, b \stackrel{\text{def}}{=} n-x \in [-cH, cH]$ . Consequently, we write

$$\sum_n f(n) \sum_m \overline{f(m)} \sum_{x \sim N} w(n-x) \overline{w(m-x)} = \sum_{|h| \in [0, 2cH]} \sum_{\substack{n \\ n-m=h}} \sum_m f(n) \overline{f(m)} \sum_{\substack{a, b \in [-cH, cH] \\ b-a=h \\ n-b=m-a \in (N, 2N]}} w(b) \overline{w(a)}.$$

Since the condition  $n-b=m-a \in (N, 2N]$  is implied by  $n, m \in (N+cH, 2N-cH]$ , then the latter is

$$\sum_{|h| \in [0, 2cH]} \sum_{\substack{n, m \in (N+cH, 2N-cH] \\ n-m=h}} \sum_m f(n) \overline{f(m)} \sum_{\substack{a, b \in [-cH, cH] \\ b-a=h}} w(b) \overline{w(a)} +$$

$$\begin{aligned}
& +O\left(H\|f\|_\infty^2 \sum_{|h|\in[0,2cH]} \left( \sum_n \sum_{\substack{m\in(N-cH,N+cH]\cup(2N-cH,2N+cH] \\ n-m=h}} 1 + \sum_m \sum_{\substack{n\in(N-cH,N+cH]\cup(2N-cH,2N+cH] \\ n-m=h}} 1 \right) \right) = \\
& = \sum_{|h|\in[0,2cH]} \left( \sum_{n\sim N} \sum_{\substack{m\in(N+cH,2N-cH] \\ n-m=h}} f(n)\overline{f(m)} + \sum_{\substack{n\in(N+cH,2N-cH], m\sim N \\ n-m=h}} f(n)\overline{f(m)} \right) \sum_{\substack{a,b\in[-cH,cH] \\ b-a=h}} w(b)\overline{w(a)} + \\
& +O\left(H\|f\|_\infty^2 \sum_{h\ll H} \sum_{\substack{m\in(N,N+cH]\cup(2N-cH,2N] \\ n-m=\pm h}} 1 \right) + O\left(H\|f\|_\infty^2 \sum_{h\ll H} \sum_{\substack{m\in(N-cH,N+cH]\cup(2N-cH,2N+cH] \\ n-m=\pm h}} 1 \right) = \\
& = \sum_{0\leq|h|\leq 2cH} \sum_{\substack{n\sim N \\ m\sim N \\ n-m=h}} f(n)\overline{f(m)} \sum_{\substack{-cH\leq a,b\leq cH \\ b-a=h}} w(b)\overline{w(a)} + O\left(H\|f\|_\infty^2 \sum_{h\ll H} \sum_{m\in(N-cH,N+cH]\cup(2N-cH,2N+cH]} 1 \right) = \\
& = \sum_{0\leq|h|\leq 2cH} \left( \mathcal{C}_f(h) + O(\|f\|_\infty^2 |h|) \right) \mathcal{C}_w(h) + O(H^2\|f\|_\infty^2(2cH+1)) = \sum_{0\leq|h|\leq 2cH} \mathcal{C}_w(h)\mathcal{C}_f(h) + O(H^3\|f\|_\infty^2) . \quad \square
\end{aligned}$$

**Remark.** The remainder term  $O(H^3\|f\|_\infty^2)$  is essentially due to the estimate of “short” segments of length  $\ll H$  within “long” sums of length  $\gg N$ . We refer to these short segments as the *tails* in the summations. In order to simplify our exposition, the symbol (T) within some of the following formulæ will warn the reader of some tails discarded to abbreviate the formulæ.

Thus, by using the *exponential sum*<sup>1</sup>

$$\widehat{f}(\beta) \stackrel{def}{=} \sum_{n\sim N} f(n)e(n\beta) ,$$

we write

$$\mathcal{C}_f(h) = \sum_{\substack{m\sim N \\ n\sim N \\ n-m=h}} f(n)\overline{f(m)} + O(\|f\|_\infty^2 |h|) = \int_0^1 |\widehat{f}(\beta)|^2 e(-h\beta) d\beta + O(\|f\|_\infty^2 |h|) \stackrel{(T)}{\approx} \int_0^1 |\widehat{f}(\beta)|^2 e(-h\beta) d\beta .$$

An important aspect is that the exponential sums, whose coefficients are correlations of a weight  $w$ , are non-negative. More precisely,

$$\widehat{\mathcal{C}}_w(\beta) = \sum_h \mathcal{C}_w(h)e(h\beta) = \sum_h \sum_{b-a=h} w(b)\overline{w(a)}e(h\beta) = \left| \sum_n w(n)e(n\beta) \right|^2 = |\widehat{w}(\beta)|^2 \quad \forall \beta \in [0,1) .$$

In particular, for the correlations of  $u = \mathbf{1}_{[1,H]}$  one gets the *Fejér kernel*

$$\widehat{\mathcal{C}}_u(\beta) = |\widehat{u}(\beta)|^2 = \left| \sum_n u(n)e(n\beta) \right|^2 = \left| \sum_{n\leq H} e(n\beta) \right|^2 .$$

More in general, the Fejér-Riesz Theorem [F] states that any non-negative exponential sum is the square modulus of another exponential sum:

$$\widehat{w}(\beta) \geq 0 \quad \forall \beta \in [0,1) \iff \exists v : \widehat{w}(\beta) = |\widehat{v}(\beta)|^2 \quad \forall \beta \in [0,1) .$$

A particularly easy instance of this theorem follows by recalling that the Cesaro weight is the normalized correlation of  $u$ , i.e.  $C_H(h) = \mathcal{C}_u(h)/H$  (see §1). Hence, again Fejér’s kernel makes its appearance in

$$\sum_h \left(1 - \frac{|h|}{H}\right)_+ e(h\beta) = \frac{1}{H} \sum_h \mathcal{C}_u(h)e(h\beta) = \frac{|\widehat{u}(\beta)|^2}{H} .$$

---

<sup>1</sup> Apart from  $\beta$  sign,  $\widehat{f}(\beta)$  is also-called the *discrete Fourier transform* of  $f$ . Hereafter we will not specify that it is a finite sum.



We also use to say that the Cesaro weights are *positive definite*. We think that basically such a property makes the aforementioned smoothing process work for the modified Selberg integral under suitable hypotheses on the function  $f$  (see §3), while for the classical Selberg integral  $J_f(N, H)$  and the symmetry integral  $J_{\text{sgn}, f}(N, H)$  it is plain that “ $u$ ” and “ $\text{sgn}$ ” are far from being positive definite weights. One could exploit such a positivity condition in order to prove a non-trivial result for the modified Selberg integral of the divisor function  $d_k$ . As a general strategy, from a non positive definite weight  $w$  with support of length at most  $H$  one could call for its normalized correlation  $\tilde{w} = \mathcal{C}_w/H$  generating the non negative exponential sum  $\widehat{\mathcal{C}_w}(\beta)/H = |\widehat{w}(\beta)|^2/H$ .

### 3. Starring: essentially bounded, balanced, quasi-constant and stable arithmetic functions.

The wide class of arithmetic functions under our consideration consists of functions bounded asymptotically by every arbitrarily small power of the variable according to the definition<sup>2</sup>:

$$f \text{ is ESSENTIALLY BOUNDED } \stackrel{\text{def}}{\iff} \forall \varepsilon > 0 \quad f(n) \ll_{\varepsilon} n^{\varepsilon} ,$$

that we shortly denote by writing  $f \ll 1$ . However, in several circumstances one has to deal with arithmetic functions having support in an interval of length  $\ll N$  or simply with functions restricted to such an interval. Thus, more in general among the essentially bounded functions we include  $f$  such that  $f(n) \ll_{\varepsilon} N^{\varepsilon} \forall \varepsilon > 0$ . A well-known prototype of an essentially bounded function is the divisor function  $d_k$ , whose Dirichlet series is  $\zeta(s)^k$ . Similarly, the Dirichlet series

$$F(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is defined at least in the right half-plane  $\Re(s) > 1$ , whenever the generating function  $f$  is essentially bounded (say the abscissa of absolute convergence is  $\sigma_{ac} \leq 1$ ). Recall that through Perron’s formula the expansion of  $F$  at  $s = 1$  leads to an asymptotic formula for the summation function from

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds \quad (x \notin \mathbb{N}) ,$$

where  $c > \max(0, \sigma_{ac})$ . More precisely, with the aid of further properties of the Dirichlet series and the Residues Theorem, such an asymptotic formula becomes

$$\sum_{n \leq x} f(n) = \text{Res}_{s=1} F(s) \frac{x^s}{s} + R_f(x) ,$$

where  $R_f(x)$  is an error term as long as it is smaller than the main term. Assuming that  $F$  is meromorphic and denoting the *polar order*<sup>3</sup> of  $F$  by  $m_F \stackrel{\text{def}}{=} \text{ord}_{s=1} F$ , the main term is more explicitly written as (compare [De])

$$xP_f(\log x) \stackrel{\text{def}}{=} \text{Res}_{s=1} F(s) \frac{x^s}{s} ,$$

where  $P_f$  is the *residual polynomial* of  $f$ , which has degree  $m_F - 1$ , while it vanishes identically when  $m_F < 1$ . For the remainder term a good estimate would be (compare [De] again)

$$R_f(x) \ll x^{\alpha(f)}$$

with a suitable  $0 \leq \alpha(f) < 1$  (negative values of exponent  $\alpha(f)$  are possible, but discarded as “meaningless”). This is the case for any divisor function  $d_k$ . Indeed, from (\*) of §0 one has

$$\sum_{n \leq x} d_k(n) = xP_{k-1}(\log x) + \Delta_k(x) \quad \text{with} \quad \Delta_k(x) \ll x^{\alpha_k} ,$$

<sup>2</sup> That is  $f$  satisfies one of the *Selberg class* axioms, the so-called *Ramanujan hypothesis* (see esp. [De]).

<sup>3</sup> That is the order of the pole of  $F$  at  $s = 1$ .

where the degree of the residual polynomial  $P_{k-1}$  (see §0) is  $k-1$ , because the polar order of  $\zeta^k$  is  $m_k = k$ , and  $\alpha_k \leq 1 - 1/k$  is what one can infer inductively from the elementary Dirichlet hyperbola method applied to the first case  $k = 2$ . More precisely, one has  $\Delta_k(x) \ll x^{1-1/k} \log^{k-2} x$ . Now, by partial summation it is easy to determine the *logarithmic polynomial*  $p_{k-1}$  such that

$$\sum_{n \leq x} p_{k-1}(\log n) = x P_{k-1}(\log x) + O(\log^{k-1} x) .$$

Thus, we get the decomposition

$$d_k(n) \stackrel{def}{=} p_{k-1}(\log n) + \widetilde{d}_k(n) ,$$

that is a “balancing” of  $d_k(n)$  from which the very slowly increasing polynomial  $p_{k-1}(\log n)$  is subtracted. The Dirichlet series generated by  $\widetilde{d}_k$  times  $x^s/s$  has zero residue at  $s = 1$ . Moreover,  $(*)$  is equivalent to

$$(*) \quad \sum_{n \leq x} \widetilde{d}_k(n) \ll x^{\alpha_k} .$$

This invites to formulate the following definitions<sup>4</sup>:

$$f \text{ is BALANCED} \iff \operatorname{Res}_{s=1} \frac{x^s}{s} F(s) = \operatorname{Res}_{s=1} \frac{x^s}{s} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = 0, \quad \forall x$$

(that is  $F$  has an analytic continuation in  $s = 1$ , because  $m_F \leq 0$ ),

$$f \text{ is WELL-BALANCED OF EXPONENT } \alpha \stackrel{def}{\iff} \sum_{n \leq x} f(n) \ll x^\alpha \text{ for some } \alpha \in [0, 1) .$$

Of course, any well-balanced function is also balanced because the previous bound implies that the Dirichlet series is regular at  $s = 1$ . However, the converse needs not to be true, as  $\Lambda(n) - 1$  is a balanced function, but the existence of an exponent  $\alpha < 1$  is still far from being proved (see some further comments below).

An essentially bounded arithmetic function  $a : \mathbb{N} \rightarrow \mathbb{C}$  is said to be **QUASI-CONSTANT** if there exists  $A \in C^1([1, +\infty), \mathbb{C})$  such that<sup>5</sup>  $A'(t) \ll 1/t$  and  $A|_{\mathbb{N}} = a$ . Clearly, the logarithmic polynomial  $p_{k-1}(\log n)$  is quasi-constant with respect to  $n$  and this, together with the fact that  $\widetilde{d}_k$  is a well-balanced arithmetic function of exponent  $\alpha_k$ , suggests the following further definition.

An arithmetic function  $f$  is **STABLE OF EXPONENT**  $\alpha$  if there exist a quasi-constant function  $a$  and a well-balanced function  $b$  of exponent  $\alpha$  such that  $f = a + b$ , while the **AMPLITUDE** of  $f$  is defined as

$$\alpha(f) \stackrel{def}{=} \inf \{ \alpha \in (0, 1) : f \text{ is stable of exponent } \alpha \} .$$

Recall that the Dirichlet divisor problem requires to prove the conjectured amplitude  $\alpha_2 = \alpha(\mathbf{d}) = 1/4$ , while one infers  $\alpha_2 \leq 1/2$  by the Dirichlet hyperbola method and the best known bound at the moment is  $\alpha_2 \leq 141/416$  (Huxley, 2003). In what follows,  $\alpha_k = \alpha(d_k)$  is the best possible exponent in  $(*)$  and  $(\widetilde{*})$ .

According to Ivić [Iv2], the mean value in the Selberg integral of any arithmetic function  $f$ , whose Dirichlet series  $F(s)$  converges absolutely at least in the half-plane  $\Re(s) > 1$  and is meromorphic in  $\mathbb{C}$ , has the *analytic form* given by

$$M_f(x, H) \stackrel{def}{=} H (P_f(\log x) + P'_f(\log x)) ,$$

<sup>4</sup> Although with a different meaning, such a terminology has been coined by Ben Green and Terence Tao. Mainly, Green [Gr] calls *balanced* a function  $f - \delta$  when  $f$  is a characteristic function of a set with density  $\delta$ .

<sup>5</sup> The condition on the derivative  $A'$  implies that  $A$  is essentially bounded, provided  $a$  depends only on  $n$ . However, we leave open the possibility that  $a$  and  $A$  might depend on auxiliary parameters.

where  $P'_f$  is the derivative of the residual polynomial of  $f$ . We remark that  $M_f(x, H)$  is linear in  $f$  and is *separable*, i.e. the variables  $H, x$  are separated. Recall that  $p_f \stackrel{def}{=} P_f + P'_f$  is the logarithmic polynomial.

Philosophically speaking, although completely justified from an analytic point of view (using mean-value theorem and derivatives bounds), such a choice of  $M_f(x, H)$  is not satisfactory, since one should expect to find the same Selberg integral for  $f$  and its balanced part  $\tilde{f} = f - p_f$ . Indeed, this is the case whenever we define the s.i. mean-value as (see §1)

$$\sum_{x < n \leq x+H} p_f(\log n) = H p_f(\log x) + O_\varepsilon(N^\varepsilon H^2/N).$$

Of course, up to negligible remainders this is still possible under Ivić's hypothesis. Let us give an idea of a possible extension of these considerations to the case of a more general function  $f$ .

Bearing in mind (\*), given any arithmetic function  $f$ , we call a polynomial  $P_f$  such that

$$(*)_f \quad \sum_{n \leq x} f(n) = x P_f(\log x) + O_\varepsilon(x^{\varepsilon+\alpha}) \quad \text{with } \alpha < 1$$

the *residual polynomial* of  $f$ , although  $P_f$  is not necessarily defined from the residues in  $s = 1$  with the Dirichlet series  $F(s)$ . Then, let us define the *logarithmic polynomial* of  $f$  as

$$p_f(\log x) \stackrel{def}{=} \frac{d}{dx} (x P_f(\log x)) = P_f(\log x) + P'_f(\log x) .$$

Under Ivić's hypothesis on  $f$  this immediately yields

$$M_f(x, H) = H p_f(\log x)$$

and

$$p_f(\log x) = \operatorname{Res}_{s=1} F(s) \frac{x^{s-1}}{s} + \operatorname{Res}_{s=1} F(s) \frac{(s-1)x^{s-1}}{s} = \operatorname{Res}_{s=1} F(s) x^{s-1} .$$

In the case  $f = d_k$  this property allows us to compare [IW] results to ours (see §0).

On the other side, by assuming the sole property  $(*)_f$  one gets a unique polynomial  $p_f$  such that

$$\sum_{n \leq x} p_f(\log n) = \int_1^x p_f(\log t) dt + O(\log^c x) = x P_f(\log x) + O(\log^c x) \sim \sum_{n \leq x} f(n) ,$$

where  $c \geq 0$  is the degree of  $p_f$ . Since  $p_f(\log x)$  is a quasi-constant function, this implies that

$$\sum_{x < n \leq x+H} f(n) \sim \sum_{x < n \leq x+H} p_f(\log n) \sim H p_f(\log x) .$$

Thus every arithmetic function  $f$  satisfying  $(*)_f$  admits a logarithmic polynomial  $p_f$  and the *analytic form* of the mean-value in short intervals, inside the  $w$ -Selberg integral, of such a function  $f$  is

$$\sum_n w(n-x) p_f(\log n) = p_f(\log x) \sum_h w(h) + O(H^2 L^{c-1}/N),$$

where  $L \stackrel{def}{=} \log N$  hereafter. In particular, the analytic form of the mean value in the Selberg integral of  $d_3$  is explicitly given by

$$M_3(x, h) = h(P_2(\log x) + P'_2(\log x)) = h\left(\frac{1}{2} \log^2 x + 3\gamma \log x + 3\gamma^2 + 3\gamma_1\right) ,$$

with  $P_2(t) = t^2/2 + (3\gamma - 1)t + (3\gamma^2 + 3\gamma_1 - 3\gamma + 1)$ , where  $\gamma$  is the Euler-Mascheroni constant and  $\gamma_1$  is a Stieltjes constant defined as

$$\gamma_1 \stackrel{\text{def}}{=} \lim_m \left( \frac{\log^2 m}{2} - \sum_{j \leq m} \frac{\log j}{j} \right).$$

Recall that this is also related to the summation formula [Ti]

$$\sum_{n \leq x} d_3(n) = xP_2(\log x) + O(x^{2/3} \log x).$$

From the application of the 3-folding method (see §7), recalling that  $x \sim N$ , it comes out that the *arithmetic form* of the mean value in the Selberg integral of  $d_3$  is (here  $M = [(N - h)^{1/3}]$ , compare §7)

$$\widetilde{M}_3(x, h) \stackrel{\text{def}}{=} h \left( \sum_{q \leq \frac{x}{M}} \frac{\mathbf{d}(q)}{q} + \sum_{d_1 < M} \frac{1}{d_1} \sum_{d_2 \leq \frac{x}{d_1 M}} \frac{1}{d_2} + \left( \sum_{d < M} \frac{1}{d} \right)^2 \right).$$

Indeed, let us show that  $M_3(x, h)$  can be replaced by  $\widetilde{M}_3(x, h)$  within  $\widetilde{J}_3(N, H)$  at the cost of a negligible error term. More precisely, we prove that

$$\widetilde{M}_3(x, h) - M_3(x, h) \ll hN^{-1/3} \quad \text{uniformly } \forall x \sim N.$$

At this aim, we apply Amitsur's formula [A] with Tull's error term [Tu]<sup>6</sup>, i.e.

$$\sum_{q \leq Q} \frac{\mathbf{d}(q)}{q} = \frac{\log^2 Q}{2} + 2\gamma \log Q + (\gamma^2 + 2\gamma_1) + O\left(\frac{1}{\sqrt{Q}}\right),$$

to get

$$i) \quad \sum_{q \leq \frac{x}{M}} \frac{\mathbf{d}(q)}{q} = \frac{1}{2} \log^2 x - (\log M) \log x + \frac{1}{2} \log^2 M + 2\gamma \log x - 2\gamma \log M + (\gamma^2 + 2\gamma_1) + O(N^{-1/3}).$$

From the standard asymptotic formula for  $\sum_{d < D} \frac{1}{d}$  one has

$$ii) \quad \sum_{d_1 < M} \frac{1}{d_1} \sum_{d_2 \leq \frac{x}{d_1 M}} \frac{1}{d_2} = (\log M + \gamma) \log x + \gamma^2 - \log^2 M - \frac{\log^2 M}{2} + \gamma_1 + O(N^{-1/3}L),$$

$$iii) \quad \left( \sum_{d < M} \frac{1}{d} \right)^2 = \log^2 M + 2\gamma \log M + \gamma^2 + O(N^{-1/3}L).$$

Thus, *i*), *ii*) and *iii*) imply the claimed inequality, because

$$\frac{\widetilde{M}_3(x, h)}{h} = \frac{1}{2} \log^2 x + 3\gamma \log x + (3\gamma^2 + 3\gamma_1) + O(N^{-1/3}L) = \frac{M_3(x, h)}{h} + O_\varepsilon(N^{\varepsilon-1/3}).$$

Such a proximity of the two terms  $\widetilde{M}_3(x, h)$  and  $M_3(x, h)$  suggests that, even for more general essentially bounded function  $f$ , one should expect the same mean value term  $M_f(x, h)$  in short intervals for the Selberg

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<sup>6</sup>Amitsur derived a symbolic method to calculate main terms of asymptotic formulæ. Tull, a student of Bateman, gave a refined partial summation, that allows here to transfer error terms from the formula for  $\sum_{q \leq Q} \mathbf{d}(q)$ , like Dirichlet's classical  $O(\sqrt{Q})$ , to this formula for  $\sum_{q \leq Q} \mathbf{d}(q)/q$ .

integral and for the modified one, whenever a suitable arithmetic form  $M_f(x, h) \approx h \sum_q g(q)/q$ , with  $f = g * \mathbf{1}$ , is proved to be sufficiently close to the analytic form (determined by the residues of the Dirichlet series generated by  $f$ ). This seems to be reliable at least when  $M_f(x, t)$  is *separable*, i.e.  $M_f(x, t) = t \mathcal{M}_f(x)$ , with  $t = o(x)$  and  $\mathcal{M}_f(x)$  is a slowly varying function with respect to  $x$  (namely, a small  $\mathcal{M}'_f(x)$  like the  $x$ -derivative of polynomials in the variable  $\log x$ ). Indeed, the identities

$$\sum_{0 \leq |n-x| \leq h} \left(1 - \frac{|n-x|}{h}\right) f(n) = \frac{1}{h} \sum_{m \leq h} \sum_{0 \leq |n-x| < m} f(n) , \quad \sum_{0 \leq |n-x| < m} f(n) \approx M_f(x, 2m-1) = (2m-1) \mathcal{M}_f(x)$$

imply together

$$\sum_{0 \leq |n-x| \leq h} \left(1 - \frac{|n-x|}{h}\right) f(n) \approx \frac{1}{h} \sum_{m \leq h} M_f(x, 2m-1) = h \mathcal{M}_f(x) = M_f(x, h) .$$

We refer the reader to the further discussion in §7 for the generalization to any divisor function  $d_k$  through the so-called *k-folding* method. Of course,  $d_k$  is not the only function suitable for  $(*)_f$ . For example, De Roton [De] has showed that, if the Dirichlet series  $F$  belongs to the so-called *Extended Selberg Class* (ESC) with<sup>7</sup>  $\deg F \geq 1$ , then  $(*)_f$  holds with  $\alpha = \frac{\deg F - 1}{\deg F + 1}$ . Further, if  $f$  is also multiplicative (so that its Dirichlet series  $F$  has a suitable Euler product) and essentially bounded, then  $F$  belongs to the special subset of ESC, called *Selberg Class* (see [KP(012)]). Hence, according to our definitions De Roton's result (after [KP(012)] breakthrough on Selberg Class) becomes:

$$f \text{ has Dirichlet series in the Selberg Class with degree } d \geq 1 \implies f \text{ is stable of exponent } \frac{d-1}{d+1} .$$

In particular, this applies to any  $d_k$ , since every power  $\zeta^k$  for  $k \geq 1$  belongs to the Selberg Class with  $\deg \zeta^k = k$ . However, the De Roton exponent  $\frac{k-1}{k+1}$  is weaker than the one obtained by other methods.

Actually, the bound  $\alpha_k = \alpha(d_k) \leq 1/2$  for  $k \leq 4$  assures that the function  $d_k$  for  $k \leq 4$  admits (at least) square-root cancellation for the error terms, a property shared by every stable arithmetic function with a sufficiently small amplitude. This motivates the following further definition:

$$f \text{ is RANDOM} \stackrel{\text{def}}{\iff} f \text{ is stable of amplitude } \alpha(f) \leq 1/2 .$$

For example, it is well-known (see [D]) that the Riemann Hypothesis (RH) is equivalent to the inequality

$$R_\Lambda(x) \stackrel{\text{def}}{=} \sum_{n \leq x} (\Lambda(n) - 1) \ll \sqrt{x} ,$$

that is to say the von Mangoldt function is random<sup>8</sup>. On the other side, it is also well-known that unconditionally the inequality  $R_\Lambda(x) \ll x^\alpha$  holds only if  $\alpha = \alpha(\Lambda) \geq 1/2$ . In other words,  $\Lambda$  cannot be stable of exponent  $\alpha < 1/2$ . Such a circumstance is better expressed by the definition:

$$f \text{ is STRICTLY RANDOM} \stackrel{\text{def}}{\iff} f \text{ is stable of amplitude } \alpha(f) = 1/2 .$$

Thus, RH is equivalent to say that prime numbers are *strictly random*. The existence of  $\alpha(\Lambda) < 1$  corresponds to a quasi-RH because of another well-known analytic property of the prime numbers<sup>9</sup>:

$$\alpha(\Lambda) = \sup \{ \beta : \zeta(\beta + i\gamma) = 0 \text{ for some } \gamma \neq 0 \} .$$

<sup>7</sup> The degree of  $F$  in ESC is defined in terms of its functional equation (see [De] for details).

<sup>8</sup> It is also a well-known fact the equivalence between RH and the *randomness* of the Möbius function  $\mu$ , suggesting that  $\mu$  behaves like  $\Lambda - 1$ . We refer to [IwKo] for further details on the *Möbius randomness law*.

<sup>9</sup> Recall also that  $\alpha(\mu) = \alpha(\Lambda)$ .

The Polya-Vinogradov inequality (see [D]) provides a non-conditional example of a strictly random arithmetic function, namely any non-principal Dirichlet character  $\chi(\bmod q)$ , since it yields

$$\sum_{n \leq q} \chi(n) \ll \sqrt{q} \log q \lll q^{1/2},$$

which is known to be essentially optimal (compare [Go] & [Te]). The actual results and the expected values  $\alpha_k = (k-1)/(2k)$  in general (see [Iv0]) reveal that the  $k$ -divisor functions are not strictly random.

Returning back to our integrals, if  $f$  is real and essentially bounded, then from Lemma 1 we have

$$\begin{aligned} J_f(N, H) &\stackrel{(\text{T})}{\sim} \sum_h \mathcal{C}_u(h) \mathcal{C}_f(h) - 2 \sum_n f(n) \sum_{x \sim N} u(n-x) M_f(x, H) + \sum_{x \sim N} M_f^2(x, H), \\ \tilde{J}_f(N, H) &\stackrel{(\text{T})}{\sim} \sum_h \mathcal{C}_{\mathcal{C}_u/H}(h) \mathcal{C}_f(h) - 2 \sum_n f(n) \sum_{x \sim N} \frac{\mathcal{C}_u(n-x)}{H} M_f(x, H) + \sum_{x \sim N} M_f^2(x, H). \end{aligned}$$

When  $f$  is also balanced, then  $M_f(x, H)$  vanishes identically. Consequently,

$$J_f(N, H) \stackrel{(\text{T})}{\sim} \sum_h \mathcal{C}_u(h) \mathcal{C}_f(h) \quad \text{and} \quad \tilde{J}_f(N, H) \stackrel{(\text{T})}{\sim} \sum_h \mathcal{C}_{\mathcal{C}_u/H}(h) \mathcal{C}_f(h).$$

#### 4. Story: smoothing correlations by arithmetic means.

Recalling that

$$\mathcal{C}_f(h) \stackrel{(\text{T})}{\sim} \int_0^1 |\hat{f}(\beta)|^2 e(-h\beta) d\beta \quad \text{with} \quad \hat{f}(\beta) = \sum_{N < n \leq 2N} f(n) e(n\beta),$$

one easily infers

$$\begin{aligned} (I) \quad \sum_h u(h) \mathcal{C}_f(h) &= \int_0^1 |\hat{f}(\beta)|^2 \hat{u}(-\beta) d\beta + O(H^2 \|f\|_\infty^2) \stackrel{(\text{T})}{\sim} \int_0^1 |\hat{f}(\beta)|^2 \hat{u}(-\beta) d\beta \\ (II) \quad \sum_h \mathcal{C}_u(h) \mathcal{C}_f(h) &= \int_0^1 |\hat{f}(\beta)|^2 \widehat{\mathcal{C}_u}(-\beta) d\beta + O(H^3 \|f\|_\infty^2) \stackrel{(\text{T})}{\sim} \int_0^1 |\hat{f}(\beta)|^2 |\hat{u}(\beta)|^2 d\beta \\ (III) \quad \sum_h \mathcal{C}_{\mathcal{C}_u/H}(h) \mathcal{C}_f(h) &= \int_0^1 |\hat{f}(\beta)|^2 \frac{|\widehat{\mathcal{C}_u}(\beta)|^2}{H^2} d\beta + O(H^3 \|f\|_\infty^2) \stackrel{(\text{T})}{\sim} \int_0^1 |\hat{f}(\beta)|^2 |\hat{u}(\beta)|^2 \cdot \frac{|\hat{u}(\beta)|^2}{H^2} d\beta \end{aligned}$$

In particular, for every balanced real function  $f \lll 1$ , from the previous section we get

$$J_f(N, H) \stackrel{(\text{T})}{\sim} \int_0^1 |\hat{f}(\beta)|^2 |\hat{u}(\beta)|^2 d\beta \quad \text{and} \quad \tilde{J}_f(N, H) \stackrel{(\text{T})}{\sim} \int_0^1 |\hat{f}(\beta)|^2 \cdot \frac{|\hat{u}(\beta)|^4}{H^2} d\beta.$$

Formulae (I), (II) and (III) correspond respectively to the following iterations of correlations' averages.

$$\text{1ST GENERATION :} \quad \sum_{h \leq H} \mathcal{C}_f(h) = \sum_h u(h) \mathcal{C}_f(h) \quad (\text{sums of correlations})$$

$$\text{2ND GENERATION :} \quad \sum_{h \leq H} \sum_{0 \leq |a| < h} \mathcal{C}_f(a) = \sum_h \mathcal{C}_u(h) \mathcal{C}_f(h) \quad (\text{double sums})$$

$$\text{3RD GENERATION :} \quad \sum_h \sum_{h_2 - h_1 = h} \frac{\mathcal{C}_u(h_1) \mathcal{C}_u(h_2)}{H^2} \mathcal{C}_f(h) = \sum_h \mathcal{C}_{\mathcal{C}_u/H}(h) \mathcal{C}_f(h) \quad (\text{average of double sums})$$

Such an obstinate process of averaging is motivated by the fact that it is rarely possible to find an asymptotic formula for the single correlation  $\mathcal{C}_f$  when  $f$  is a significant arithmetic function. As we already said, the correlation of  $f$  counts the number of  $h$ -twins not only when  $f$  is a pure characteristic function (the von Mangoldt function is a typical case). In general, the underlying Diophantine equation is a binary problem that is out of reach with the present methods (see next §5 for some details). On the other side, the higher is the degree of a generation of the correlations' averages, the smoother are such averages and consequently we have more hope to get non-trivial asymptotic estimates. However, even at a 2nd generation level this hope is quite frustrated by the lack of efficient elementary methods<sup>10</sup> to bound directly the Selberg integral. Indeed, Ivić [Iv2] applies Riemann zeta moments since the Selberg integral of  $d_k$  has a strong connection with them (see §8). Further, it is interesting to analyze the cost of the loss when a non-trivial information on the correlations' averages at some  $n$ th generation level is transferred to the averages of the  $(n-1)$ th generation. For example, assuming that  $f$  is real, essentially bounded and balanced, from the trivial inequality  $\widehat{u} \ll H$  one immediately has

$$\widetilde{J}_f(N, H) \ll J_f(N, H) + H^3.$$

Then, in order to obtain an inequality in the opposite direction, we appeal to the formulæ deduced from (II) and (III) and write

$$J_f(N, H) \ll \int_0^1 |\widehat{f}(\beta)|^2 |\widehat{u}(\beta)|^2 d\beta + H^3 \ll \sqrt{N \int_0^1 |\widehat{f}(\beta)|^2 |\widehat{u}(\beta)|^4 d\beta} + H^3 \ll \sqrt{NH^2 \widetilde{J}_f(N, H)} + \sqrt{N} H^{5/2} + H^3,$$

where we have applied the Cauchy inequality and the Parseval identity (with  $f \ll 1$ ). Thus, if for some  $H$  one has a non-trivial estimate of the kind

$$\widetilde{J}_f(N, H) \ll NH^2/G$$

with some gain  $G \rightarrow \infty$ , then the previous formula implies (for the same range of  $H$ )

$$J_f(N, H) \ll NH^2 G^{-1/2} + N^{1/2} H^{5/2}.$$

This gives Theorem 1 by taking  $G = H$  in Conjecture CL.

Hence, by the sole application of the Cauchy inequality a third generation gain  $G \ll N/H$  leads to the gain  $\sqrt{G}$  for the second generation estimate. We say that the *exponent's gain has halved*.

The same phenomenon occurs for a general weight  $w$  with the alternative approach that we describe here assuming  $\widetilde{J}_{w,f}(N, H) \ll NH^2/G$ . By taking  $E \stackrel{\text{def}}{=} \{\beta : |\widehat{w}(\beta)| \geq \varepsilon H\}$  we have that the following

$$\begin{aligned} \int_0^1 |\widehat{f}(\beta)|^2 |\widehat{w}(\beta)|^2 d\beta &\ll \varepsilon^2 H^2 \int_{[0,1] \setminus E} |\widehat{f}(\beta)|^2 d\beta + \frac{1}{\varepsilon^2} \int_E |\widehat{f}(\beta)|^2 \cdot \frac{|\widehat{w}(\beta)|^4}{H^2} d\beta \\ &\ll NH^2 \|f\|_\infty^2 \varepsilon^2 + \frac{1}{\varepsilon^2} \widetilde{J}_{w,f}(N, H) \ll NH^2 \|f\|_\infty^2 (\varepsilon^2 + G^{-1} \varepsilon^{-2}) \end{aligned}$$

is an optimal bound when  $\varepsilon = G^{-1/4}$  and we get the same halving of the exponent's gain as before.

Now, recalling that  $|\widehat{u}(\beta)| \gg H$  when  $|\beta| < 1/(2H)$  (see [D], Ch.25), we get also

$$H^2 \int_{-\frac{1}{2H}}^{\frac{1}{2H}} |\widehat{f}(\beta)|^2 d\beta \ll \int_{-\frac{1}{2H}}^{\frac{1}{2H}} |\widehat{f}(\beta)|^2 \cdot \frac{|\widehat{u}(\beta)|^4}{H^2} d\beta \leq \int_0^1 |\widehat{f}(\beta)|^2 \cdot \frac{|\widehat{u}(\beta)|^4}{H^2} d\beta = \widetilde{J}_f(N, H) + O_\varepsilon(N^\varepsilon H^3),$$

that is a *modified* version of Gallagher's Lemma [Ga, Lemma 1]. In order to establish the aforementioned estimate of  $J_3$  the first author in [C5] applies such a version of Gallagher's Lemma together with the following

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<sup>10</sup> This will be coped by our forthcoming paper on mean-squares in short intervals ( $w$ -Selberg integrals).

further property of the essentially bounded and balanced functions; namely (see [C5] for the proof), if, for an absolute constant  $A \in [0, 1)$  and for a fixed  $\delta \in (0, 1/2)$  one has  $N^\delta \ll H_1 \ll H_2 \ll N^{1/2-\delta}$  and

$$\tilde{J}_f(N, H) \ll NH^{1+A}, \quad \forall H \in [H_1, H_2], \quad \text{then } J_f(N, H) \ll NH^{1+\frac{1+3A}{5-A}}, \quad \forall H \leq H_2.$$

What about the trade of information from the second generation to the first?

Let us take  $f = a + b$  real and essentially bounded, with  $b$  balanced. Then, we write

$$\begin{aligned} \sum_{h \leq H} \mathcal{C}_f(h) &\stackrel{(T)}{\sim} \sum_{h \leq H} \sum_{n \sim N} f(n)f(n+h) = \sum_{x \sim N} f(x) \sum_{x < m \leq x+H} f(m) = \\ &= \sum_{x \sim N} f(x) \sum_{x < m \leq x+H} a(m) + \sum_{x \sim N} f(x) \sum_{x < m \leq x+H} b(m). \end{aligned}$$

Again by the Cauchy inequality one has

$$\sum_{x \sim N} f(x) \sum_{x < m \leq x+H} b(m) \ll \left( \sum_{x \sim N} |f(x)|^2 \sum_{x \sim N} \left| \sum_{x < n \leq x+H} f(n) - M_f(x, H) \right|^2 \right)^{1/2} \ll N^{1/2} J_f(x, H)^{1/2},$$

where the mean value is

$$M_f(x, H) \stackrel{def}{=} \sum_{x < n \leq x+H} a(n).$$

Hence, we get exactly an asymptotic formula with main term

$$\sum_{x \sim N} f(x) M_f(x, H),$$

whenever the tails are negligible and mostly the remainder term

$$\sum_{x \sim N} f(x) \sum_{x < m \leq x+H} b(m) \ll N^{1/2} J_f(x, H)^{1/2}$$

turns out to be sufficiently small after halving the exponent's gain on  $J_f(x, H)$ .

Moreover, when  $a$  is quasi-constant, one has  $M_f(x, H) \stackrel{(T)}{\sim} a(x)H$ , that implies

$$\begin{aligned} \sum_{h \leq H} \mathcal{C}_f(h) &\stackrel{(T)}{\sim} H \sum_{x \sim N} a(x)^2 + H \sum_{x \sim N} a(x)b(x) + O_\varepsilon(N^{1/2+\varepsilon} J_f(x, H)^{1/2}) \stackrel{(T)}{\sim} \\ &\stackrel{(T)}{\sim} H \int_N^{2N} a(t)^2 dt + H \sum_{x \sim N} a(x)b(x) + O_\varepsilon(N^{1/2+\varepsilon} J_f(x, H)^{1/2}). \end{aligned}$$

In particular, if  $f$  is stable of exponent  $\alpha$ , then by applying partial summation to  $\sum_{x \sim N} a(x)b(x)$  one definitively gets the asymptotic inequality for the *deviation* of  $f$ , say,

$$\mathbb{D}_f(N, H) \stackrel{def}{=} \sum_{h \leq H} \mathcal{C}_f(h) - H \sum_{x \sim N} a(x)^2 \ll NHG^{-1/4} + N^\alpha H + H^2,$$

whenever a non-trivial estimate for the second generation,  $J_f(x, H) \ll NH^2G^{-1/2}$ , holds for some ranges of short intervals width, large enough in terms of the exponent  $\alpha$ .

Hence, we find a possible general chain of implications of non-trivial estimates as

$$\tilde{J}_f(N, H) \ll NH^2G^{-1} \implies J_f(N, H) \ll NH^2G^{-1/2} \implies \mathbb{D}_f(N, H) \ll NHG^{-1/4}.$$



The exponent's gain halves at each step, but if it remains a neat positive one, then we say that  $f$  is *stable through generations*. In §6 complete calculations are supplied for the case of the divisor function  $d_3$ , while in future papers we are going to explore the hardest case of stable arithmetic functions (eventually) having no logarithmic polynomial.

## 5. Toccata e fuga: the rare cases of asymptotic formulæ for correlations.

An expected asymptotic formula for the single correlations of  $f$  usually takes the form (compare [BP])

$$\mathcal{C}_f(h) = \mathfrak{S}_f(h)\mathcal{I}_f(N) + \mathcal{R}_f(N, h) ,$$

where the product of the so-called *singular series*  $\mathfrak{S}_f$  and the *singular integral*  $\mathcal{I}_f$  constitutes the main term, while  $\mathcal{R}_f(N, h)$  has to be proved an error term. Such terminology is customary within the Circle Method, that was originally introduced by Hardy, Littlewood and Ramanujan in 1918-20 to attack several additive Diophantine problems. One of the most famous and maddening problem is the infinitude of the pairs of  $2h$ -*twin* primes  $(n, n - 2h)$  that can be formulated in terms of the correlation of the von Mangoldt function:

$$\mathcal{C}_\Lambda(2h) = \sum_{n \sim N} \Lambda(n)\Lambda(n - 2h)$$

Indeed, with the aid of the powerful analytic tools, Hardy and Littlewood predicted that for every sufficiently large  $N$  one has

$$\mathcal{C}_\Lambda(2h) = \mathfrak{S}_\Lambda(2h)N + \mathcal{R}_\Lambda(N, 2h)$$

with a certain singular series  $\mathfrak{S}_\Lambda(2h) \gg 1$  and a conjectured remainder term  $\mathcal{R}_\Lambda(N, 2h) \ll N(\log N)^{-A}$  for every constant  $A > 0$ . Note that such an asymptotic formula would imply the infinitude of the  $2h$ -twin primes, but nowadays nobody has yet proved such a conjecture.

Apart from very special cases of functions or some trivial instances of the correlations (as they could be when  $h = 0$ ), the lack of asymptotic formulæ involves the correlations of the most significant arithmetic functions in number theory. One of the exceptional cases is given by the well-known *binary additive divisor problem*,

$$\mathcal{C}_\mathbf{d}(a) = \sum_{n \sim N} \mathbf{d}(n)\mathbf{d}(n - a)$$

where  $\mathbf{d}$  is the divisor function. This problem was known at least since [E] time and has been studied mainly through the consolidated theory of modular forms on  $SL(\mathbb{Z}, 2)$  (see [IwKo], [Vi]). By adapting Motohashi's results [Mo] to the problem in short intervals, namely  $0 \neq a = o(N)$ , one has

$$\mathcal{C}_\mathbf{d}(a) = \mathfrak{S}_\mathbf{d}(a)N + \mathcal{R}_\mathbf{d}(N, a) ,$$

where

$$\mathfrak{S}_\mathbf{d}(a) = \mathfrak{S}_\mathbf{d}(a, \log N) \stackrel{def}{=} \sum_{i=0}^2 (\log N)^i \sum_{j=0}^2 c'_{i,j} \sum_{d|a} \frac{\log^j d}{j} ,$$

$$\mathcal{R}_\mathbf{d}(N, a) \ll N^{2/3} + N^{1/2}|a|^{9/40} + |a|^{7/10}$$

(here these absolute  $c'_{i,j}$  are not Motohashi's constants [Mo, p.530]). In search of remarkable improvements on the latter formula, one has to be content with an extensive literature on moments of  $\mathcal{R}_\mathbf{d}(N, a)$  (see [IM]). The general case of the additive divisor problem for  $d_k$ , i.e. establishing an asymptotic formula for

$$\mathcal{C}_k(a) \stackrel{def}{=} \sum_{n \sim N} d_k(n)d_k(n - a) ,$$

is much harder and still unsolved when  $k \geq 3$ . Even the basic case of  $\mathcal{C}_3(a)$  seems to be hopeless with present technology due to the poor knowledge about the structure of  $SL(\mathbb{Z}, 3)$ . We address the interested reader to [C2] for a very short tale about this fascinating problem. Moreover, we recall that the Dispersion Method of

Linnik (see his book [L]) made its appearance to attack these kind of binary problems. In §2 we show only an elementary version of the Dispersion Method, while we have to mention the beautiful paper [Iw] as an example of a non-trivial application of it.

In the modular forms environment the correlations are widely known as *Shifted Convolution Sums* (see [Mi]). In such a context, the possibility of establishing asymptotic formulæ depends directly on the same structure of the modular forms. This is particularly successful for the special class of arithmetic functions given by the Hecke eigenvalues  $\lambda(n)$ . For example, Conrey and Iwaniec [CoIw] provide an asymptotic estimate

$$\mathcal{C}_\lambda(h) \sim N \sum_{q=1}^{\infty} c_q(h) p(q)^2 ,$$

where  $c_q(h)$  is a Ramanujan sum, while we refer the reader to the quoted paper for the definition of  $p(q)$  and the intricate remainder term. It is remarkable that in a joint and unpublished work with Iwaniec the first author has easily deduced from the aforementioned formula of [CoIw] non-trivial bounds for the symmetry integral of the eigenvalues  $\lambda$  (essentially square-root cancellation). Other spectacular advances on bounding the correlations of the Hecke eigenvalues have been achieved by Holowinsky in [Ho1], [Ho2] and then applied jointly with Soundararajan in [HoSo].

Returning to more familiar functions as  $d_k$ , it seems that the only hope remains the Large Sieve and all the methods strictly related to such an inequality. In fact, similarly to the binary additive divisor problem, the Large Sieve turns out to be crucial when one looks for asymptotic formulæ for “*mixed*” correlations of  $\mathbf{d}$  with some “reasonable” multiplicative arithmetic function (see [C0], where the fundamental reference is Linnik’s book [L]). Essentially the reason is that, due to the Dirichlet hyperbola method, the divisor function has “*level*”  $\ell < 1/2$  of distribution in the arithmetic progressions, which is the same “*large-sieve-barrier*” for primes in arithmetic progressions, i.e. the celebrated Bombieri-Vinogradov Theorem. Note that the barrier  $\ell < 1/2$  prevents one from applying this very strong tool to the correlations of the von Mangoldt function  $\Lambda$ , but it is harmless with the “truncated von Mangoldt” function  $\Lambda_R$ , used in the breakthrough of Goldston-Pintz-Yildirim on small gaps between primes [GPY]. An alternate approach appeals to Duke-Friedlander-Iwaniec [DFI] bounds for bilinear forms of Kloosterman fractions instead of the Bombieri-Vinogradov theorem. It allowed the first author [C] to successfully establish asymptotic formulæ for the correlations of essentially bounded functions  $f$  such that the Eratosthenes transform  $f * \mu$  is supported up to  $O(x^{\frac{1}{2} + \frac{1}{190} - \varepsilon})$  and  $\Lambda_R(n)$  with  $R \ll x^{\frac{1}{2} + \frac{1}{190} - \varepsilon}$  might be a remarkable example. However, this is a small improvement on the “level” for correlations, with respect to the aforementioned level in the arithmetic progressions given by the Large Sieve barrier. See the book [El] for the links between the concepts of *level*.

A further possibility is open when  $f * \mu$  vanishes outside of a very sparse set, where the “low density” has an actual effect on the level in arithmetic progressions. In this direction, we refer the reader to [BPW] and [To] on the  $k$ -free numbers, whose characteristic function is defined by

$$\sum_{d^k | n} \mu(d) = \sum_{q|n} g_k(q) .$$

Indeed, it is plain that here the Eratosthenes transform  $g_k$  is supported on the  $k$ -th powers, a very “low-density” support.

## 6. Crescendo: asymptotic formulæ for $d_3$ in almost all short intervals.

Here we turn our attention to non-trivial bounds for the Selberg integral of  $d_3$ , postponing the general discussion on  $d_k$  to next section. Ivić [Iv2] proved that the inequality

$$J_3(N, H) \ll N^{1-\varepsilon_1} H^2$$

holds for the width  $\theta > 1/6$  with a neat exponent’s gain  $\varepsilon_1 > 0$ . In other words, defining  $\theta > \theta_3$  as the range of the admissible width of the short interval for such an inequality, Ivić has proven  $\theta_3 = 1/6$ . This is built upon the value  $\sigma_3 \leq 7/12$ , where  $\sigma_k$  is the so-called *Carlson’s abscissa* for the Riemann zeta  $2k$ -th moment,

$$\sigma_k \stackrel{def}{=} \inf \{ \sigma \in [0, 1] : \int_1^T |\zeta(\sigma + it)|^{2k} dt \ll T \}$$

(see next section for some of the known values  $\sigma_k$  quoted from [Iv0]).

Conjecture CL provides improvements on Ivić's result since for every width  $\theta \leq 1/3$  it yields the “best possible estimate”, i.e. the square-root cancellation  $\tilde{J}_3(N, H) \ll NH$ , which in turn implies the optimal  $\theta_3 = 0$  through the arguments of §4. This estimate allows improvements on recent results [BBMZ] and [IW] for sums of correlations of  $d_3$ . Further, it is worth to recall here that the lower bound  $J_3(N, H) \gg NH \log^4 N$  holds if  $0 < \theta < 1/3$  (see [C0]).

Now let us prove Corollary 1.

PROOF OF COROLLARY 1. From the decomposition  $d_3(n) = p_2(\log n) + \tilde{d}_3(n)$  introduced in §3 one gets

$$\begin{aligned} \mathcal{C}_3(h) &\stackrel{\text{def}}{=} \sum_{n \sim N} d_3(n) d_3(n-h) = \sum_{N+h < m \leq 2N+h} d_3(m+h) d_3(m) = \sum_{n \sim N} d_3(n+h) d_3(n) + O_\varepsilon(N^\varepsilon |h|) = \\ &= \sum_{n \sim N} \tilde{d}_3(n) \tilde{d}_3(n+h) + 2 \sum_{n \sim N} \tilde{d}_3(n) p_2(\log n) + \sum_{n \sim N} p_2(\log n)^2 + O_\varepsilon(N^\varepsilon |h|), \end{aligned}$$

where recall that  $d_3, \tilde{d}_3$  are essentially bounded, while  $p_2(\log n)$  is a quasi-constant function of  $n$ . Therefore,

$$\mathbb{D}_3(N, H) = \sum_{h \leq H} \mathcal{C}_3(h) - H \sum_{n \sim N} p_2(\log n)^2 = \sum_{n \sim N} \tilde{d}_3(n) \sum_{n < m \leq n+H} \tilde{d}_3(m) + 2H \sum_{n \sim N} \tilde{d}_3(n) p_2(\log n) + O_\varepsilon(N^\varepsilon H^2).$$

By applying partial summation and  $(*)$  one has

$$\sum_{n \sim N} \tilde{d}_3(n) p_2(\log n) = p_2(\log(2N)) \sum_{n \sim N} \tilde{d}_3(n) - \int_N^{2N} \sum_{N < n \leq t} \tilde{d}_3(n) \frac{p'_2(\log t)}{t} dt \ll \max_{t \leq 2N} \left| \sum_{N < n \leq t} \tilde{d}_3(n) \right| \ll N^{\alpha_3}.$$

Since the lower bound  $J_3(N, H) \gg NH \log^4 N$  holds at least for width  $0 < \theta < 1/3$  (see [C0]), then Cauchy's inequality implies

$$\sum_{n \sim N} \tilde{d}_3(n) \sum_{n < m \leq n+H} \tilde{d}_3(m) \ll \sqrt{N \left( \sum_{x \sim N} \left| \sum_{x < m \leq x+H} d_3(m) - M_3(x, H) \right|^2 + \frac{H^4 L^2}{N} \right)} \ll \sqrt{N J_3(N, H)},$$

where recall that  $L = \log N$  and (see §3)

$$M_3(x, H) = H p_2(\log x) = \sum_{x < m \leq x+H} p_2(\log m) + O(x^{-1} H^2 L).$$

Thus, the conclusion follows immediately from Theorem 1.  $\square$

## 7. Main Theme: from all long intervals to almost all short intervals.

The inductive identity  $d_k = d_{k-1} * \mathbf{1}$  invites to explore a possible path in order to generalize our conjectures and results to each divisor function  $d_k$  for  $k > 3$  by inferring formulæ for  $d_k$  in almost all short intervals from suitable information on  $d_{k-1}$  in long intervals. However, the actual known values of the amplitudes  $\alpha_k$  seem to be a first serious bottle-neck. Further, while it might be comparatively easy to attack Conjecture CL, the path climbs up drastically when it comes to the general case  $k > 3$ . Although a general  $k$ -folding method is available (see next Lemma 2, that is essentially the core of such a method), more and more technical problems are foreseeable as  $k$  increases. Besides, at the outset one has to face the problem of showing sufficient proximity of the analytic and the arithmetic forms of the mean value, at least by mean-square approximation, i.e. an inequality of the form

$$\sum_{x \sim N} \left| M_k(x, H) - \widetilde{M}_k(x, H) \right|^2 \ll N^{1-2/k} H^2,$$

for a suitable choice of  $\widetilde{M}_k(x, H)$ . Being unconceivable to give a rigorous general proof of such a proximity by direct calculations as we did for the case  $k = 3$  in §3, at the moment we have to content ourself with the following heuristic considerations. Some results of Ivić [Iv2] provide non-trivial estimates of the Selberg integral  $J_k(N, H)$  with the mean value  $M_k(x, H)$  assigned in the appropriate analytic form. Hence, Theorem 2 legitimates the assumption that the arithmetic mean value  $\widetilde{M}_k(x, H)$  is close to the analytic counterpart  $M_k(x, H)$  for every  $k > 3$  in the ranges of the short interval width  $\theta$  provided by Ivić's results. Actually, we know that the analytic form is  $H$  times a  $k - 1$  degree polynomial in  $\log x$ , the same shape that approximates (see next Lemma 2) the arithmetic form. Then, comparing Ivić's results with ours in a common range for  $\theta$ , we easily conclude that these polynomials must coincide (Amitsur formula [A] gives polynomials' degree and Tull's Lemma [Tu] the remainders).

Now, let us turn our attention to next Lemma 2, where we show the so-called “ $k$ -folding method”. At this aim, assuming that  $H$  does not depend on  $x$  explicitly, let us consider for any fixed  $k \geq 2$  the weighted sum

$$S_k(x, H) \stackrel{def}{=} \sum_n a_k(n) w(n - x) = \sum_{n_1} \cdots \sum_{n_k} a(n_1) \cdots a(n_k) w(n_1 \cdots n_k - x),$$

where  $a_k \stackrel{def}{=} \underbrace{a * \cdots * a}_{k \text{ times}}$  is the  $k$ -fold Dirichlet product of an arithmetic function  $a : \mathbb{N} \rightarrow \mathbb{C}$  and the weight  $w : \mathbb{N} \rightarrow \mathbb{C}$  is supposed to be uniformly bounded in its support, that is contained in  $[-H, H]$ .

Further, for  $M \stackrel{def}{=} [(N - H)^{1/k}]$  let us denote

$$g_k(q) = g_k(q, a, M) \stackrel{def}{=} a_{k-1}(q) + \sum_{j \leq k-1} a_{k-1}^{(j)}(q),$$

where

$$a_{k-1}^{(j)}(q) \stackrel{def}{=} \sum_{\substack{n_1 \cdots n_{k-1} = q \\ n_1, \dots, n_j < M}} a(n_1) \cdots a(n_{k-1}) \quad \forall j \leq k - 1.$$

LEMMA 2. *If the arithmetic function  $a$  is essentially bounded, then  $\forall \varepsilon > 0$  and uniformly for every  $x \sim N$ ,*

$$S_k(x, H) = \sum_{q \leq x/M} g_k(q) \sum_{\substack{0 \leq |n-x| \leq H \\ n \equiv 0 \pmod{q}}} a\left(\frac{n}{q}\right) w(n - x) + O_{k, \varepsilon} \left( N^\varepsilon \left( \frac{H}{N^{1/k}} + \frac{H^2}{N} + 1 \right) \right).$$

PROOF. First note that  $n_1 \cdots n_k \geq x - H \geq N - H$  implies that at least one of  $n_1, \dots, n_k$  has to be  $\geq M$ . Then, let us define the partial sums  $\Sigma_0, \Sigma_1, \dots, \Sigma_{k-1}$  of  $S_k(x, H)$  as follows:

$\Sigma_0$  is the part of  $S_k(x, H)$  corresponding to  $n_1 \geq M$ ,

$\Sigma_1$  is the part of  $S_k(x, H) - \Sigma_0$  corresponding to  $n_2 \geq M$ ,

$\Sigma_2$  is the part of  $S_k(x, H) - \Sigma_0 - \Sigma_1$  corresponding to  $n_3 \geq M$ , and so on.

Therefore, we set  $a_{k-1}^{(0)}(q) \stackrel{def}{=} a_{k-1}(q)$  and split  $S_k(x, H)$  as

$$\begin{aligned} S_k(x, H) &= \Sigma_0 + \Sigma_1 + \dots + \Sigma_{k-1} = \\ &= \sum_{q \leq \frac{x+H}{M}} a_{k-1}^{(0)}(q) \sum_{\substack{n=qm \\ m \geq M}} a(m) w(n - x) + \sum_{q \leq \frac{x+H}{M}} a_{k-1}^{(1)}(q) \sum_{\substack{n=qm \\ m \geq M}} a(m) w(n - x) + \\ &+ \dots + \sum_{q \leq \frac{x+H}{M}} a_{k-1}^{(k-1)}(q) \sum_{\substack{n=qm \\ m \geq M}} a(m) w(n - x) = \sum_{q \leq \frac{x+H}{M}} g_k(q) \sum_{\substack{n=qm \\ m \geq M}} a(m) w(n - x), \end{aligned}$$

where for each  $j = 0, 1, \dots, k - 1$  the multiple sum

$$a_{k-1}^{(j)}(q) = \sum_{n_1} \cdots \sum_{\substack{n_{k-1} \\ n_1 \cdots n_{k-1} = q \\ n_1, \dots, n_j < M}} a(n_1) \cdots a(n_{k-1})$$

has  $j$  variables restricted by  $M$  (that depends on  $N, H, k$ , but not on  $x$ ).

Observe that, since  $g_k(q) = \sum_{j=0}^{k-1} a_{k-1}^{(j)}(q)$ , then  $|g_k(q)| \leq k|a|_{k-1}(q) \ll 1$ , where we set  $|a|_{k-1} \stackrel{\text{def}}{=} \underbrace{|a| * \dots * |a|}_{k-1 \text{ times}}$ .

Thus,

$$\begin{aligned}
S_k(x, H) &= \sum_{q \leq \frac{x+H}{M}} g_k(q) \sum_{\substack{\frac{x-H}{q} \leq m \leq \frac{x+H}{q} \\ m \geq M}} a(m)w(qm-x) = \\
&= \sum_{q \leq \frac{x-H}{M}} g_k(q) \sum_{\frac{x-H}{q} \leq m \leq \frac{x+H}{q}} a(m)w(qm-x) + \mathcal{O}_{k,\varepsilon} \left( \sum_{\frac{x-H}{M} < q \leq \frac{x+H}{M}} \sum_{|m-\frac{x}{q}| \leq \frac{H}{q}} x^\varepsilon \right) = \\
&= \sum_{q \leq x/M} g_k(q) \sum_{\frac{x-H}{q} \leq m \leq \frac{x+H}{q}} a(m)w(qm-x) + \mathcal{O}_{k,\varepsilon} \left( \sum_{\frac{x-H}{M} < q \leq \frac{x+H}{M}} \sum_{|m-\frac{x}{q}| \leq \frac{H}{q}} x^\varepsilon \right) = \\
&= \sum_{q \leq x/M} g_k(q) \sum_{\substack{0 \leq |n-x| \leq H \\ n \equiv 0 \pmod{q}}} a\left(\frac{n}{q}\right) w(n-x) + \mathcal{O}_{k,\varepsilon} \left( N^\varepsilon \sum_{\frac{x-H}{M} < q \leq \frac{x+H}{M}} \sum_{|m-\frac{x}{q}| \leq \frac{H}{q}} 1 \right).
\end{aligned}$$

Since  $q > \frac{x-H}{M} \gg \frac{N}{M}$  uniformly as  $N \leq x \leq 2N$ , then the  $\mathcal{O}_{k,\varepsilon}$ -term is

$$\ll N^\varepsilon \sum_{\frac{x-H}{M} < q \leq \frac{x+H}{M}} \left( \frac{H}{q} + 1 \right) \ll N^\varepsilon \left( \frac{H}{M} + 1 \right) \left( \frac{HM}{N} + 1 \right) \ll N^\varepsilon \left( \frac{H^2}{N} + \frac{H}{N^{1/k}} + 1 \right). \quad \square$$

An application of Lemma 2 to attack the General Conjecture CL would require  $a = \mathbf{1}$ , so that  $a_k = d_k$ , attached to the Cesaro weights.

**Remark.** The General Conjecture CL together with Theorem 2 justifies the title of the present section: we positively interpret the (upper) bounds for the amplitude  $\alpha_{k-1}$  as a property which holds in every long interval, while the consequent bound of  $\tilde{J}_k$  clearly means an almost all short interval property. However, since the present state of knowledge assigns values to  $\alpha_k$  that grow towards 1 with  $k$ , the quality of the General Conjecture CL and Theorem 2 ruins unrelentingly for large values of  $k$ . Of course, as already seen when  $k = 3$ , the general scheme of implications of non-trivial estimates discussed in §4 still works in suitable ranges of short intervals, namely when the width of the short interval is sufficiently large with respect to  $\alpha_k$ . In particular, one easily proves Corollary 2 on the deviation of  $d_k$  (see the definition in §0) for any  $k > 3$  by closely following the Corollary 1 proof with the aid of the arguments in §4.

Note that we have an improvement on the aforementioned Ivić's results on the Selberg integral  $J_k(N, H)$  for  $k > 3$ . More precisely, Theorem 1 in [Iv2] holds for  $\theta \in (\theta_k, 1)$  with  $\theta_k \stackrel{\text{def}}{=} 2\sigma_k - 1$ . In particular, from Corollary 2 of [Iv2] one has

$$\theta_4 = \frac{1}{4}, \quad \theta_5 = \frac{11}{30}, \quad \theta_6 = \frac{3}{7},$$

respectively corresponding to the following known upper bounds [Iv0] for Carlson's abscissae, defined in §6,

$$\sigma_4 \leq \frac{5}{8}, \quad \sigma_5 \leq \frac{41}{60}, \quad \sigma_6 \leq \frac{5}{7}.$$

Due to Corollary 2 such values of  $\theta_k$  are superseded by

$$\tilde{\theta}_4 = \frac{11}{128}, \quad \tilde{\theta}_5 = \frac{1}{5}, \quad \tilde{\theta}_6 = \frac{1}{3},$$

which follow from the well-known upper bounds for the amplitudes (see [Ti])

$$\alpha_3 \leq \frac{43}{96} \text{ (Kolesnik, 1981), } \quad \alpha_4 \leq \frac{1}{2} \text{ (Hardy-Littlewood [HL], 1922), } \quad \alpha_5 \leq \frac{11}{20} \text{ ([Iv0], Ch.13) .}$$

It might be possible that  $\theta_k < \tilde{\theta}_k$  for any  $k > 6$ , although we think that it is unlikely. However, we put our emphasis essentially on the method: unlike Ivić, we do not use any deep property of the Riemann  $\zeta$ -function (at least not explicitly) and rely uniquely upon known values of the exponents  $\alpha_k$ ; but one should not forget that most known values  $\alpha_k$  follow directly from non-trivial estimates of the moments of the Riemann zeta function on the line (for a clear digression on such a topic see the wonderful book by Ivić [Iv0]). On the other side, as the first author pointed out (see [C2], Theorem 1.1), estimates of the Selberg integral of  $d_k$  have non-trivial consequences on the  $2k$ -th moments of the Riemann zeta function. Thus, recalling that Kolesnik has found his bound for  $\alpha_3$  without the aid of the 6-th moments and that  $\alpha_k$  known values go to 1, it is plain that, at least for relatively low values of  $k$ , Theorem 2 confirms that the Selberg integrals of  $d_k$  and the  $2k$ -th moments of the Riemann zeta function are connected by a circle route. Next section is devoted to a further discussion on the argument.

### 8. Finale: conditional bounds for the moments of $\zeta$ on the critical line.

As already mentioned in the previous section, at least in theory we could draw some consequences of Theorem 2 on the moments of the Riemann zeta function. However, our method does not lead to a better result than those available in the literature for the  $2k$ -th moment when  $4 \leq k \leq 6$  and maybe the scenario is even worst when  $k > 6$ . The reason is essentially the bound  $N^{1-1/k}H^2$  in the General Cojecture CL, that is an unavoidable barrier term. It transfers from  $\tilde{J}_k(N, H)$  to a bound of the  $2k$ -th moment, via the Selberg integral  $J_k(N, H)$ , whenever we appeal to Theorem 1.1 of [C2].

Here we take the opportunity of applying Theorem 1.1 of [C2] to give some conditional bounds, depending on estimates for the Selberg integrals  $J_k$  which are proved for  $k = 3$ , but unproved for  $k > 3$ . At this aim, let us define the *excess*  $E_k$  as a real number such that

$$I_k(T) = \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \ll T^{1+E_k}.$$

With an application of Hölder's inequality from the well-known values  $E_2 = 0$  and  $E_6 = 1$  (from Heath-Brown's bound  $I_6(T) \ll T^2$ ) one gets

$$E_3 = 1/4, \quad E_4 = 1/2, \quad E_5 = 3/4.$$

According to this definition, Theorem 3 gives  $E_k = \frac{k}{2}(A+B) - B$  whenever  $J_k(N, H) \ll N^{1+A}H^{1+B}$  holds for  $H \ll N^{1-2/k}$ . In particular, as already mentioned, it gives  $E_3 = 1/10$ , when combined with recent [C5].

PROOF OF THEOREM 3. In [C2] the Selberg integral of  $d_k$  is defined in the standard way as

$$\int_{hx^\varepsilon}^x \left| \sum_{t < n \leq t+h} d_k(n) - M_k(t, h) \right|^2 dt.$$

It is easy to see that a dyadic argument allows to replace  $[hx^\varepsilon, x]$  by dyadic intervals like  $[N, 2N]$ , and the substitution of the integral on  $[N, 2N]$  by  $J_k(N, H)$  generates only negligible remainder terms. Hence, we may apply Theorem 1.1 of [C2] by using  $J_k(N, H)$  instead of the above integral and  $\forall \varepsilon > 0$  small we get

$$\begin{aligned} I_k(T) &\ll T + T \max_{T^{1+\varepsilon} \ll N \ll T^{\frac{k}{2}}} \frac{T}{N^2} \max_{0 < H \ll \frac{N}{T}} J_k(N, H) \ll T + T \max_{T^{1+\varepsilon} \ll N \ll T^{\frac{k}{2}}} N^A \left( \frac{N}{T} \right)^B \ll \\ &\ll T + T \left( T^{\frac{k}{2}} \right)^A \left( T^{\frac{k}{2}-1} \right)^B \ll T^{1+\frac{k}{2}(A+B)-B}. \quad \square \end{aligned}$$

### 9. Epilogo: the best unconditional exponent of $|\zeta|^6$ mean on the line.

As we saw in the introduction, a further immediate consequence of the recent result [C5] is that we have an elementary deduction of  $E_3 = 1/10$  just taking  $A = 0$  and  $B = 1/5$  in Theorem 3 for  $H \ll N^{1/3}$ . However, the proof of Theorem 1.1 [C2] uses the approximate functional equation for  $\zeta^k$  (and Gallagher's Lemma),

so the whole study actually proving  $E_3 = 1/10$  (in [C5] Gallagher's Lemma is also applied) is just shorter, rather than elementary, with respect to the Heath-Brown's bound of  $I_6$ .

Furthermore, see Corollary of [C5], our new approach based on the "modified Gallagher Lemma", contained in the forthcoming paper [CL], assures under Conjecture CL the even better excess  $E_3 = 0$  for the Riemann zeta function, namely the well-known *weak 6-th moment*.

The excesses  $E_k$  here ( $k \geq 3$ ) are from a 2nd generation approach, while Ivić's [Iv1] is a 1st generation one.

## REFERENCES

- [A] Amitsur, S. A. - *Some results on arithmetic functions* - J. Math. Soc. Japan **11** (1959), 275–290. [MR 26#67](#) - available online
- [BBMZ] Baier, S., Browning, T.D., Marasingha, G. and Zhao, L. - *Averages of shifted convolutions of  $d_3(n)$*  - <http://arxiv.org/abs/1101.5464v2>
- [BP] Brüdern, J., Perelli, A. - *A note on the distribution of sumsets* - Funct. Approx. Comment. Math. **29** (2001), 81–88. [MR 2005m:11033](#)
- [BPW] Brüdern, J., Perelli, A. and Wooley, T. - *Twins of  $k$ -Free Numbers and Their Exponential Sum* - Michigan Math. J. **47** (2000), No. 1, 173–190. [MR 2001c:11113](#) - available online
- [CoIw] Conrey, B. and Iwaniec, H. - *Spacing of zeros of Hecke  $L$ -functions and the class number problem* - Acta Arith. **103** (2002), no. 3, 259–312. [MR 2003h:11103](#)
- [C] Coppola, G. - *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals, III* - <http://arxiv.org/abs/1003.0302v1>
- [C0] Coppola, G. - *On some lower bounds of some symmetry integrals* - <http://arxiv.org/abs/1003.4553v2> - to appear on Afrika Matematika (Springer)
- [C1] Coppola, G. - *On the modified Selberg integral* - <http://arxiv.org/abs/1006.1229v1>
- [C2] Coppola, G. - *On the Selberg integral of the  $k$ -divisor function and the  $2k$ -th moment of the Riemann zeta-function* - Publ. Inst. Math. (Beograd) (N.S.) **88(102)** (2010), 99–110. - available online
- [C3] Coppola, G. - *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals, II* - Int. J. Pure Appl. Math. **58.3**(2010), 281–298. - available online
- [C4] Coppola, G. - *On the symmetry of divisor sums functions in almost all short intervals* - Integers **4** (2004), A2, 9 pp. (electronic). [MR 2005b:11153](#)
- [C5] Coppola, G. - *On the Selberg integral of the three-divisor function  $d_3$*  - available online at the address <http://arxiv.org/abs/1207.0902v3>
- [CL] Coppola, G. and Laporta, M. - *A modified Gallagher's Lemma* - <http://arxiv.org/abs/>
- [CS] Coppola, G. and Salerno, S. - *On the symmetry of the divisor function in almost all short intervals* - Acta Arith. **113** (2004), no. 2, 189–201. [MR 2005a:11144](#)
- [D] Davenport, H. - *Multiplicative Number Theory* - Third Edition, GTM 74, Springer, New York, 2000. [MR 2001f:11001](#)
- [De] De Roton, A. - *On the mean square of the error term for an extended Selberg class* - Acta Arith. **126** (2007), no. 1, 27–55. [MR 2007j:11121](#)
- [DFI] Duke, W., Friedlander, J. and Iwaniec, H. - *Bilinear forms with Kloosterman fractions* - Invent. Math. **128** (1997), no. 1, 23–43. [MR 97m:11109](#)
- [E] Estermann, T. - *Über die Darstellungen einer Zahl als Differenz von zwei Produkten* - J. Reine Angew. Math. (Crelle Journal) **164** (1931), 173–182.
- [El] Elliott, P.D.T.A. - *On the correlation of multiplicative and the sum of additive arithmetic functions* - Mem. Amer. Math. Soc. **112** (1994), no. 538, viii+88 pp. [MR 95d:11099](#)
- [F] Fejér, L. - *Über trigonometrische Polynome* - J. für Math. (Crelle Journal) **146** (1916), 53–82.
- [Ga] Gallagher, P. X. - *A large sieve density estimate near  $\sigma = 1$*  - Invent. Math. **11** (1970), 329–339. [MR 43#4775](#)
- [Go] Goldmakher, L. - *Character sums to smooth moduli are small* - Canad. J. Math. **62** (2010), no. 5, 1099–1115. [MR 2011k:11108](#)
- [Gr] Green, B. - *On arithmetic structures in dense sets of integers* - Duke Math. J. **114** (2002), no. 2, 215–238. [MR 2003i:11021](#)
- [GPY] Goldston, D.A., Pintz, J. and Yıldırım, C. - *Primes in tuples. I* - Ann. of Math. (2) **170** (2009), no. 2, 819–862. [MR 2011c:11146](#)

- [HL] Hardy, G.H. and Littlewood, J.E. - *The approximate functional equation in the theory of the zeta-function, with applications to the divisor problems of Dirichlet and Piltz* - Proc. London Math. Soc.(2) **21** (1922), 39–74.
- [Ho1] Holowinsky, R. - *A sieve method for shifted convolution sums* - Duke Math. J. **146** (2009), no. **3**, 401–448. [MR2010b:11127](#)
- [Ho2] Holowinsky, R. - *Sieving for mass equidistribution* - Ann. of Math. (2) **172** (2010), no. **2**, 1499–1516. [MR2011i:11060](#)
- [HoSo] Holowinsky, R. and Soundararajan, K. - *Mass equidistribution for Hecke eigenforms* - Ann. of Math. (2) **172** (2010), no. **2**, 1517–1528. [MR2011i:11061](#)
- [Iv0] Ivić, A. - *The Riemann Zeta Function* - John Wiley & Sons, New York, 1985. (2nd ed., Dover, Mineola, N.Y. 2003). [MR87d:11062](#)
- [Iv1] Ivić, A. - *The general additive divisor problem and moments of the zeta-function* - New trends in probability and statistics, Vol. **4** (Palanga, 1996), 69–89, VSP, Utrecht, 1997. [MR99i:11089](#)
- [Iv2] Ivić, A. - *On the mean square of the divisor function in short intervals* - J. Théor. Nombres Bordeaux **21** (2009), no. **2**, 251–261. [MR2010k:11151](#)
- [IM] Ivić, A. and Motohashi, Y. - *On some estimates involving the binary additive divisor problem*, Quart. J. Math. Oxford Ser. (2) **46**, no. **184** (1995), 471–483. [MR96k:11117](#)
- [IW] Ivić, A. and Wu, J. - *On the general additive divisor problem* - <http://arxiv.org/abs/1106.4744v2>
- [Iw] Iwaniec, H. - *Almost-primes represented by quadratic polynomials* - Invent. Math. **47** (1978), no. **2**, 171–188. [MR58#5553](#)
- [IwKo] Iwaniec, H. and Kowalski, E. - *Analytic Number Theory* - American Mathematical Society Colloquium Publications, 53. AMS, Providence, RI, 2004. [MR2005h:11005](#)
- [KP] Kaczorowski, J. and Perelli, A. - *On the distribution of primes in short intervals* - J. Math. Soc. Japan **45** (1993), no. **3**, 447–458. [MR94e:11100](#)
- [KP(012)] Kaczorowski, J. and Perelli, A. - *On the structure of the Selberg class, VII:  $1 < d < 2$*  - Ann. of Math. (2) **173** (2011), no. **3**, 1397–1441.
- [L] Linnik, Ju.V. - *The Dispersion Method in Binary Additive Problems* - Translated by S. Schuur - American Mathematical Society, Providence, R.I. 1963. [MR29#5804](#)
- [Mi] Michel, P. - *On the Shifted Convolution Problem* - available online at the following web address, <http://tan.epfl.ch/files/content/sites/tan/files/PhMICHELfiles/Fields2003.pdf>
- [Mo] Motohashi, Y. - *The binary additive divisor problem* - Ann. Sci. École Norm. Sup. (4) **27** (1994), no. **5**, 529–572. [MR95i:11104](#)
- [S] Selberg, A. - *On the normal density of primes in small intervals, and the difference between consecutive primes* - Arch. Math. Naturvid. **47** (1943), no. **6**, 87–105. [MR7,48e](#)
- [Te] Tenenbaum, G. - *Introduction to Analytic and Probabilistic Number Theory* - Cambridge Studies in Advanced Mathematics, **46**, Cambridge University Press, 1995. [MR97e:11005b](#)
- [Ti] Titchmarsh, E. C. - *The theory of the Riemann zeta-function* - Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986. [MR88c:11049](#)
- [To] Tolev, D. - *On the distribution of  $r$ -tuples of square-free numbers in short intervals* - Int. J. Number Theory **2** (2006), no. **2**, 225–234. [MR2008a:11111](#)
- [Tu] Tull, J. P. - *Average order of arithmetic functions* - Illinois J. Math. **5** (1961), 175–181. [MR22#10943](#)
- [Vi] Vinogradov, A. I. - *A generalized square of the zeta function. Spectral decomposition* (Russian) - Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **322** (2005), Trudy po Teorii Chisel, 17–44, 251; translation in J. Math. Sci. (N. Y.) **137** (2006), no. **2**, 4617–4633. [MR2006g:11175](#)

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